Homotopy equivalence in graph-like digital spaces

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Introduction

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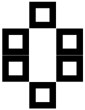

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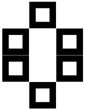
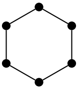


This can be abstracted to a study of simple graphs using the tools of digital topology.



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Definition

Let X, Y be digital spaces. We say $f : X \to Y$ is continuous if whenever $x_1 \leftrightarrow x_2$ in X, $f(x_1) \leftrightarroweq f(x_2)$ in Y.

Homotopy Equivalence

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For $f, g : X \to Y$, a *homotopy* from f to g is a map $H : X \times [0, n]_{\mathbb{Z}} \to Y$ such that $H(x, 0) = f(x), H(x, n) = g(x)$, and $H(x, t)$ is continuous in both x and t.

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Definition

X and Y are *homotopy equivalent* if there exist two continuous functions $f : X \to Y, \quad g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, and $g \circ f \simeq \mathrm{id}_X$.

Examples

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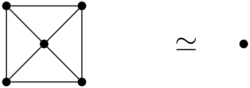
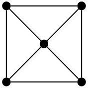
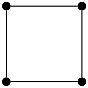


Examples

 $\simeq$ •

 $\simeq$ •

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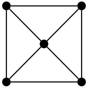
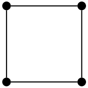
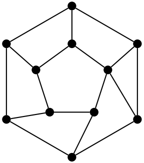
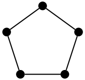
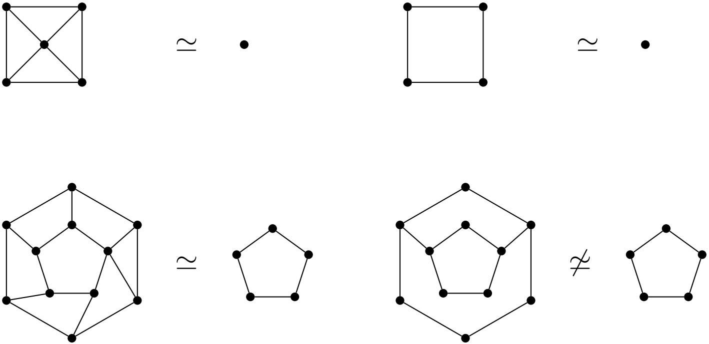
 $\simeq$ 

 $\simeq$ 

 $\simeq$ 

Examples



Homotopy Invariants

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A property of a space X is a *homotopy invariant* if $X \simeq Y$ implies Y also has this property.

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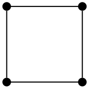
Classically, we use the Euler characteristic.



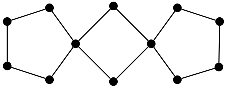
$\chi(X) = 4 - 4 = 0$

$\chi(X) = 1$

Homotopy Invariants

Homotopy Invariants

$L_m(X)$ is the number of simple, irreducible m-loops in X that are distinct up to homotopy.



$L_4(X) = 0$ $L_4(X) = 0$

$L_5(X) = 4$ $L_5(X) = 4$

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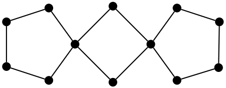
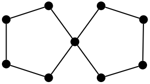


$A_4(X) = 8$
$A_5(X) = 20$



$A_4(X) = 0$
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Cataloging Irreducible Spaces

How many connected spaces exist on n points that are distinct up to homotopy equivalence?

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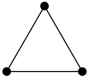
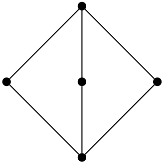
Points	1	2	3	4	5	6	7	8	9
Spaces	1	1	2	6	21	112	853	11,117	261,080

Theorem

If there exists a complete subgraph $S \subseteq X$ such that all points are adjacent to S, then X is reducible.

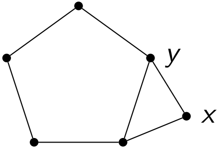




Techniques and Tools

Theorem

If there exists $x, y \in X$ such that the neighborhood of x is contained in the neighborhood of y then X is reducible.



Catalog

Points	1	2	3	4	5	6	7	8	9
Before	1	1	2	6	21	112	853	11,117	261,080
After	1	0	0	0	1	2	15	160	3,251

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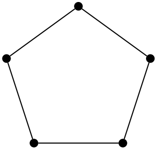
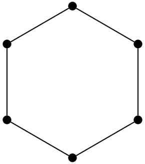
$n = 1$:

$n = 5$:



$n = 6$:



$n = 7$: