Hook Sets of Partitions Corresponding to Numerical Semigroups

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Background

Definition

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The even numbers $\{x \in \mathbb{N} \mid x = 2k,\ k \in N\}$ are **not** a numerical semigroup, since their complement is infinite.

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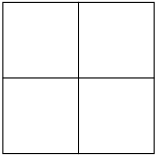

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2	1

Each box in the Young diagram has a *hook length*, the number of boxes directly to the right of this box, plus the number of boxes directly below it, plus one for the box itself.

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In the Young diagram for $(2,2)$, $H(\lambda) = \{1,2,3\}$

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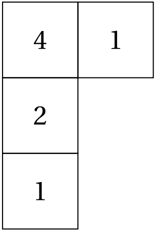

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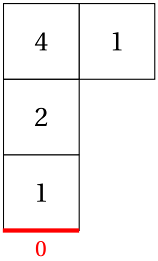

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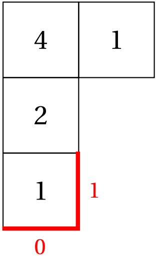

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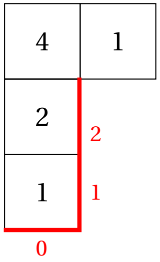

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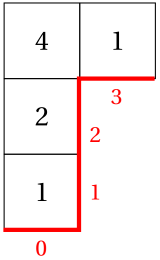

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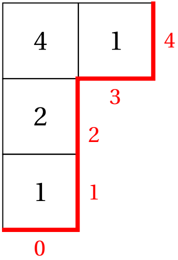

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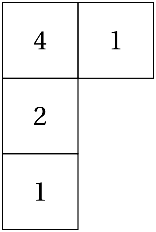

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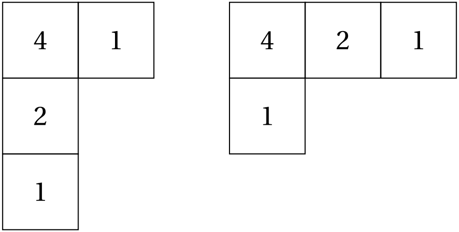

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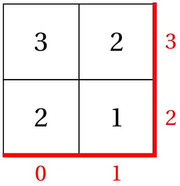

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The hook set of any partition uniquely determines a semigroup S, and thus $\varphi(S)$.

Motivating question

Fix a semigroup S. Given the hook set $H(\varphi(S)) = \mathbb{N} \setminus S$, define P_S to be the number of partitions with hook set $= H(\varphi(S))$.

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For given S, what is P_S?

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Furthermore, $H(\lambda) = \{n - m \mid n \notin T, m \in T, n > m\}$.

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A semigroup is called **pseudo-symmetric** if F is even, and the only exception to the condition of symmetry is the case $a = F/2$.

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Proposition

$P_S = 1$ if and only if S is a symmetric semigroup. Furthermore, if S is pseudo-symmetric, then P_S is always 2, and the partitions are $\varphi(S)$ and $\varphi(S)^T$, its conjugate.

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Thus, $P_S = 1 \iff S$ is symmetric.

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Then by translating the results of A. Marzuola and J. Miller, combined with a result of J. Backelin, we have the following proposition.

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Acknowledgments

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