Graph associahedra and Hassett spaces

Rodrigo Ferreira da Rosa

Introduction

The space $M_{0,n}$ of configurations of n distinct points on $\mathbb{P}^1$ is a ubiquitous object in algebraic geometry. Hassett produces a family $\overline{M}_{0,\omega}$ of compactifications of $M_{0,n}$ by allowing points to collide in varying degrees, based on their "weight" ω.

A *graph associahedron* $\mathcal{P}G$ is a convex polytope induced by a finite graph G. There is a toric variety $X(\mathcal{P}G)$ associated to $\mathcal{P}G$, and an intersting question with combinatorial and geometric implications is for which G, $X(\mathcal{P}G)$ is isomorphic to a Hassett space.

A classic example is the Losev–Manin compactification of $M_{0,n}$, which is given by $\omega = (1, 1, \epsilon, \epsilon, ..., \epsilon)$ and is isomorphic to $X(\mathcal{P}K_{n-2})$, the toric variety associated to graph associahedron of the complete graph on $n-2$ vertices.

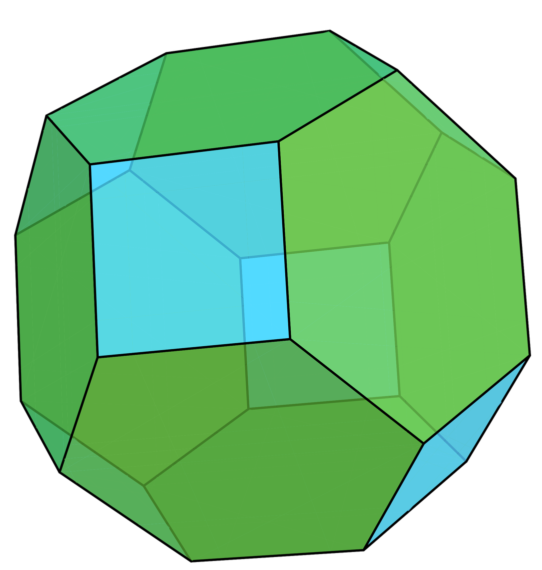

Figure 1: The permutohedron $\mathcal{P}K_4$

We completely classify all Hassett spaces that arise as the toric varieties of graph associahedra.

Graph associahedra

Given a graph G on $n-2$ vertices, fix a bijection between the vertices of G and the facets of the $(n-3)$-simplex Δ.

Definition: A *tube* t of G is a subset of the vertices of G with connected induced subgraph. t naturally corresponds to the face of Δ given by the intersection of the facets in bijection to the vertices in t.

Construction of $\mathcal{P}G$: Find all tubes of G and truncate the corresponding faces of Δ in increasing order of dimension.

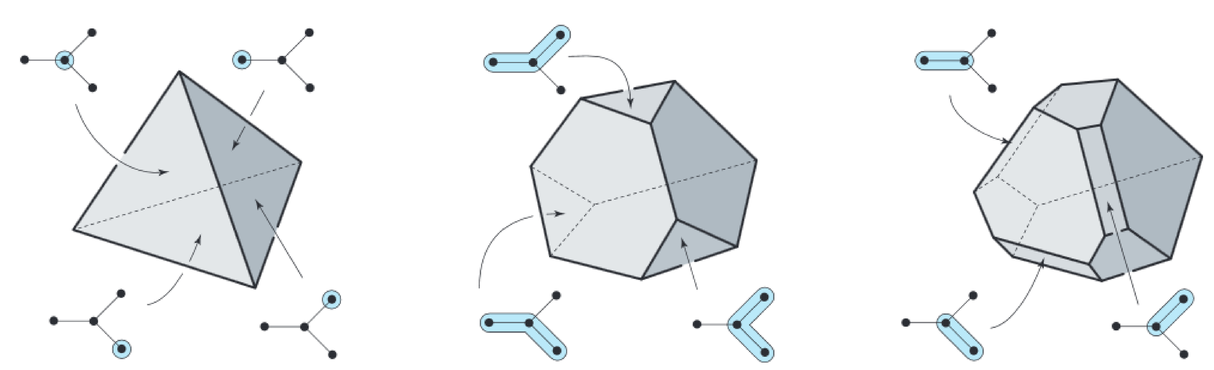

Figure 2: The truncation of the stellahedron

Compactifications of $M_{0,n}$

The coordinates planes of $\mathbb{P}^n$ are in bijection with the faces of the n-simplex. Given a graph G, $X(\mathcal{P}G)$ is defined to be the blowup of $\mathbb{P}^n$ (in increasing order of dimension) along the subvarieties corresponding to truncated faces of $\mathcal{P}G$. In order to understand $X(\mathcal{P}G)$ as a compactification of $M_{0,n}$, we use ideas due to Kapranov, and the following classical fact:

Theorem: There exists a unique rational normal curve through any n general points of $\mathbb{P}^{n-3}$.

Of the set of n points, let the toric fixed points of $\mathbb{P}^{n-3}$ be the last $n-2$ points, $NT = [1 : ... : 1]$ be the second point, and a moving point m be the first point.

Whenever m is not inside the span of $n-2$ other points, there exists a unique rational normal curve through all points, parametrizing a $\mathbb{P}^1$ with n marked points. If m collides with the span V of $k \leq n-2$ of the other points, we get a nodal degeneration of $\mathbb{P}^1$ where the n marked points either collide or bubble off depending on whether or not V was blown up.

Hassett spaces

Hassett defines $\overline{M}_{0,\omega}$ by assigning weights to the n marked points. Heuristically, "light points" may collide, while "heavy points" may not.

Definition: Given a weight vector $\omega = (c_m, c_{NT}, c_1, ..., c_{n-2}) \in [0,1]^n$, $\overline{M}_{0,\omega}$ is the space of all genus zero rational nodal curves, all of whose components are ω-stable. A component Λ is ω-stable if:

$$\#\text{ of nodes} + \sum_{i \in \Lambda} c_i > 2$$

A Hassett space is a compactification of $M_{0,n}$ of the form $\overline{M}_{0,\omega}$ for some ω.

Results

Main Theorem: If $X(\mathcal{P}G) = \overline{M}_{0,\omega}$ for some ω, then $G = \text{Cone}^{n-2-k}(\sqcup_{i=1}^{k} v_i)$.

Example: If G is a star graph, then $X(\mathcal{P}G) \cong \overline{M}_{0,\omega}$ for $\omega = (1, 1/2, 1/2, \epsilon, \epsilon, ..., \epsilon)$.

The proof of the main theorem is based on the following observations.

Proposition: If G is a graph, then there exists a weight vector ω such that $\overline{M}_{0,\omega}$ is isomorphic to $X(\mathcal{P}G)$ iff the following inequalities can be satisfied:

1. for every tube t, $|t| > 1$, $c_{NT} + \sum_{i \in t} c_i > 1$
2. for every non-tube ν, $c_{NT} + \sum_{i \in \nu} c_i \leq 1$.
3. $\left(\sum_{i=1}^{n-2} c_i \right) - c_i \leq 1, \forall c_i \neq c_m, c_{NT}$

Obstruction 1: If $X(\mathcal{P}G)$ is a Hassett space, no non-tube $\nu \subset G$ contains a tube t with $|t| > 1$.

Obstruction 2: If $X(\mathcal{P}G)$ is a Hassett space, there is no $S \subset V(G)$ s.t. S can be partitioned in k tubes and k' non-tubes with $k \geq k'$.

Corollary (of obstruction 2): If $X(\mathcal{P}\,\text{Susp}(G))$ is a Hassett space, G is the complete graph.

Advisor: Dave Jensen, Dhruv Ranganathan