The Jacobian group of a finite graph and the monodromy pairing

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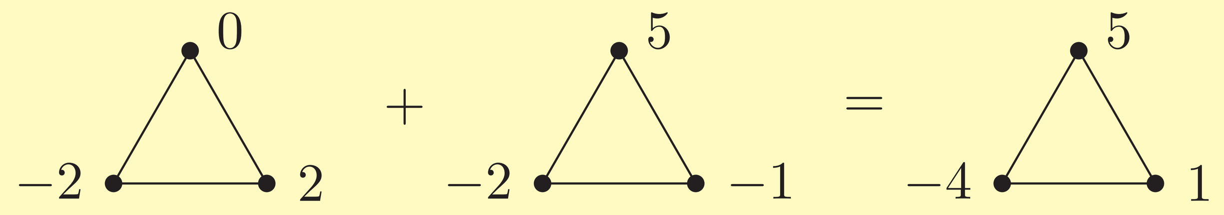
The Jacobian group of a graph

Divisors on a graph

Let G be a finite, connected multigraph with no loop edges. Let $V(G), E(G)$ be the vertex and edge sets of G.

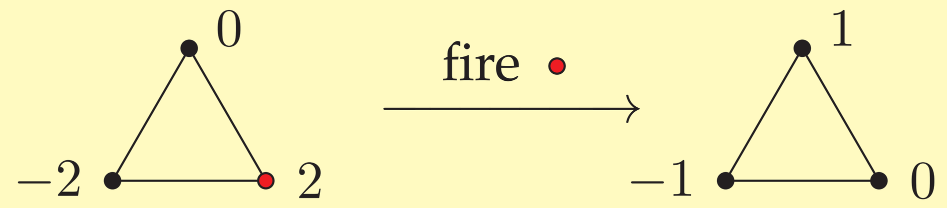
- a **divisor** is an element of $\mathrm{Div}(G) = \bigoplus_{v \in V(G)} \mathbb{Z}(v)$, the free abelian group generated by $V(G)$

We think of divisors as "configurations of chips" on the vertices of G, with group structure given by pointwise addition:



Chip-firing

You can "fire" a vertex on a divisor to get a new divisor:



If you can get between two divisors D and D' via a sequence of chip-firing moves, then D, D' are **equivalent**, $D \sim D'$.

The Jacobian group

- $\deg(D) =$ the total number of chips in D
- $\mathrm{Div}^0(G) = \{D \in \mathrm{Div}(G) : \deg(D) = 0\}$

Now we define the **Jacobian** group of the graph G:

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\sim .$$

For any connected graph G, $\mathrm{Jac}(G)$ is a finite abelian group.

Fact. $|\mathrm{Jac}(G)| =$ the number of spanning trees of G.

This fact is a consequence of the Matrix Tree Theorem.

Pairings on finite abelian groups

Given a finite abelian group Γ, a **pairing** on Γ is an inner-product like structure $\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$ that is *symmetric*, *bilinear*, and *non-degenerate*.

Orthogonal sum

Given two groups Γ_1, Γ_2 with pairings $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$, the **orthogonal sum** is a natural pairing defined on $\Gamma_1 \times \Gamma_2$ by

$$\langle (a_1, a_2), (b_1, b_2) \rangle = \langle a_1, b_1 \rangle_1 + \langle a_2, b_2 \rangle_2.$$

Pairings on $\mathbb{Z}/p^r\mathbb{Z}$

If p is an odd prime, then, up to isomorphism, there are precisely two pairings on $\mathbb{Z}/p^r\mathbb{Z}$. Written additively, they are given by

$$\langle x, y \rangle = xy/p^r \quad \text{and} \quad \langle x, y \rangle = axy/p^r,$$

where a is some quadratic nonresidue modulo p^r. We call these the **residue** and **nonresidue** pairings, respectively.

Classification theorem

Given any pairing on any finite abelian group of *odd* order, we can write it as the orthogonal sum of pairings on $\mathbb{Z}/p^r\mathbb{Z}$ terms.

The monodromy pairing on the Jacobian

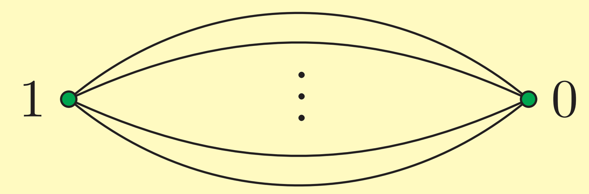
Given a graph G, $\mathrm{Jac}(G)$ is endowed with a canonical pairing called the **monodromy pairing**. Thus it is natural to ask:

Question. Which groups with pairing appear as the Jacobians of finite graphs?

We obtain the orthogonal sum of two Jacobians with pairing by taking the wedge sum of the corresponding graphs. Hence it suffices to find graphs with $\mathrm{Jac}(G) \simeq \mathbb{Z}/p^r\mathbb{Z}$ and with each of the two pairing isomorphism classes as listed above.

The residue pairing

Consider B_{p^r}, the **banana graph** (fig. 1) on p^r edges:



(fig. 1)

For every p and r, $\mathrm{Jac}(B_{p^r}) \simeq \mathbb{Z}/p^r\mathbb{Z}$, and the pairing on $\mathrm{Jac}(B_{p^r})$ is isomorphic to the residue pairing.

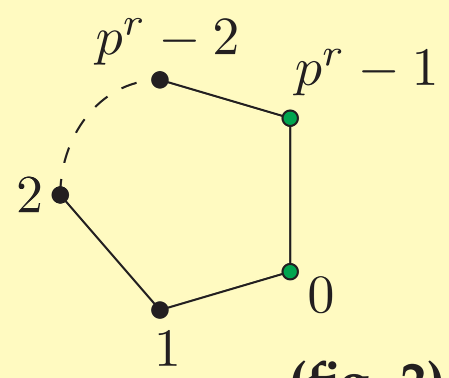
The nonresidue pairing

A candidate for getting the nonresidue pairing is the **cycle graph** C_{p^r} (fig. 2). Indeed $\mathrm{Jac}(C_{p^r}) = \mathbb{Z}/p^r\mathbb{Z}$, and the pairing is isomorphic to $\langle x, y \rangle = (-1)xy/p^r$.

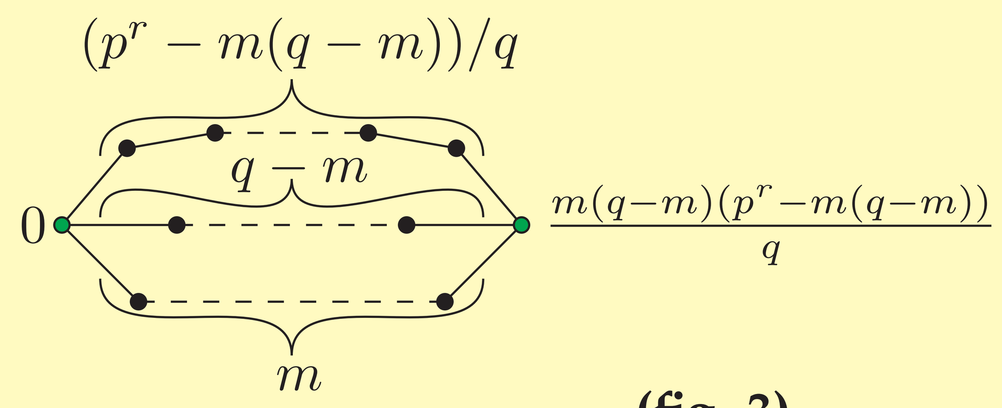
However, it is not true that -1 is always a nonresidue modulo p^r. This is only true when $p \equiv 3 \pmod 4$, so we must consider another construction when $p \equiv 1 \pmod 4$. For this, the following claim is sufficient.

Claim. There exists q, a nonresidue mod p, such that $q^2/4 < p^r$ and $q \equiv 3 \pmod 4$ if r is odd and $q \equiv 1 \pmod 4$ if r is even.

Given this claim, we pick m such that $p^r \equiv m(q-m) \pmod q$, and construct the following **subdivided banana graph** (fig. 3):



(fig. 2)



(fig. 3)

This construction gives us the nonresidue pairing on $\mathbb{Z}/p^r\mathbb{Z}$ with $p \equiv 1 \pmod 4$, provided that the above claim holds. For $r \geq 2$, we have a proof of this claim. In the case that $r = 1$, the result is implied by the **Generalized Riemann Hypothesis**.

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