Non-Left-Orderable Surgeries on Twisted Torus Knots

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REU mini-conference at Yale, 2014

Outline

Left Orderability

Definition (Left Orderable)

A non-trivial group G is left orderable if it admits a strict total ordering $<$ on its elements that is left invariant, i.e. if $g < h$, then $fg < fh$ for all $f \in G$.

- Ex: $(\mathbb{Z}, +), <$
- Non-Ex: $(\mathbb{Z}_m, +)$

Definitions

Definition (Heegaard Floer Homology)

Heegaard Floer homology is a 3-manifold invariant which associates an $\mathbb{F}_2$-vector space to a closed 3-manifold.

Definition (L-Space)

A closed, connected, orientable 3-manifold is an L-Space if it has the "simplest possible" Heegaard Floer homology.

The Boyer–Gordon–Watson L-Space Conjecture

Conjecture (Boyer–Gordon–Watson)

An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable.

Knots (1)

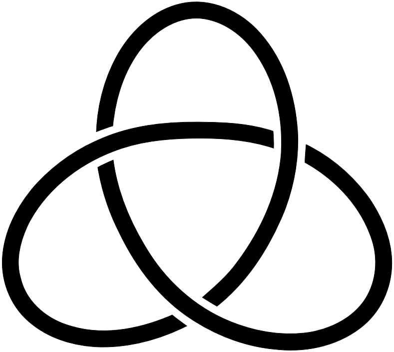

Figure : A right-handed trefoil knot.

Knots (2)

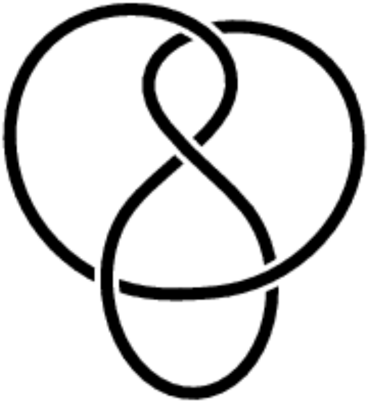

Figure : A figure-eight knot.

Torus Knots (1)

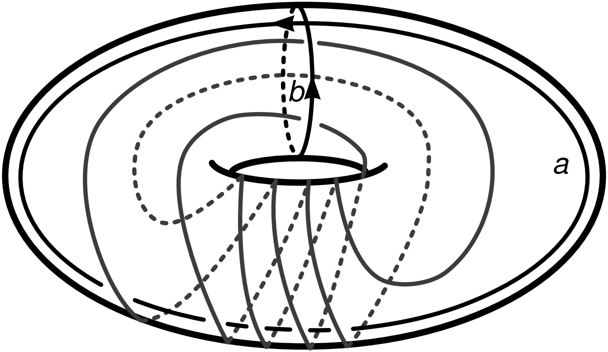

Figure : The $(3, 5)$-torus knot. (Clay–Watson)

Torus Knots (2)

Figure : The $(3, 5)$-torus knot as a braid.

Twisted Torus Knots

- $T_{p,q}^{\ell,m}$ denotes the (p, q)-torus knot with ℓ strands twisted m full times.

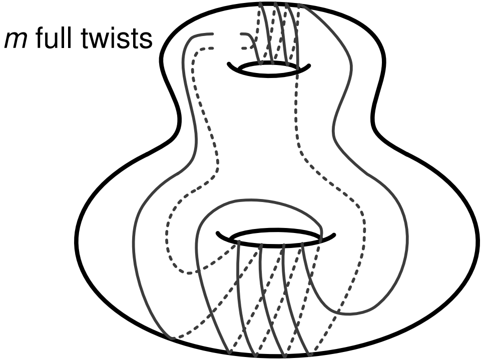

Figure : $T_{3,5}^{2,m}$ (Clay–Watson)

Dehn Surgery

Definition (Dehn Surgery)

Consider a twisted torus knot in S^3. Dehn surgery is the process of removing a neighborhood of the knot (a solid torus) from S^3 and gluing it back in. This process is specified by a rational number r.

Theorem (Vafaee)

Sufficiently large Dehn surgery performed on $T^{\ell,m}_{p,pk\pm1}$ yields an L-space for either (1) $\ell = p-1$ or (2) $m = 1$ and $\ell = p-2$ or (3) $m = 1$ and $\ell = 2$.

- Let $G^{\ell,m}_{p,q}(r)$ denote the fundamental group of the manifold that results from r-surgery on $T^{\ell,m}_{p,q}$.

Results of Clay–Watson

Theorem (Clay–Watson)

$G^{2,1}_{3,3k+2}(r)$ and $G^{2,m}_{3,5}(r)$ are not left-orderable for sufficiently large r.

- Proof involves case-by-case analysis of generator signs
- Sub-cases of Case 1 ($G^{p-1,m}_{p,pk\pm1}(r)$) in Vafaee

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Theorem 1 (KC, JG, LH, SV)

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Results (2)

Theorem 2 (KC, JG, LH, SV)

$G^{p-2,1}_{p,pk\pm1}(r)$ is not left orderable for sufficiently large r.

- Results support the L-Space Conjecture.

Characterizing Left Orderability

Theorem

A countable group G is left orderable if and only if it is isomorphic to a subgroup of $\mathrm{Homeo}^+(\mathbb{R})$.

Global Fixed Points

Definition (Global Fixed Point)

Let G be a group and let $\Phi : G \to \text{Homeo}^+(\mathbb{R})$ be a group homomorphism. Φ has a global fixed point if there exists a real number x such that $\Phi(g)x = x$ for all $g \in G$.

Proposition (Boyer–Rolfson–Wiest)

If there exists such Φ with non-trivial image, then there exists another such homomorphism which induces an action on $\mathbb{R}$ with no global fixed points.

- Suffices to show that every homomorphism $\Phi : G^{\ell,m}_{p,pk\pm1}(r) \to \text{Homeo}^+(\mathbb{R})$ has a global fixed point.

Outlook

- Lower bound on r in our results is larger than the lower bound on surgeries that yield L-spaces.
- The third case of L-spaces described by Vafaee ($m = 1$ and $\ell = 2$) remains unresolved.