Tropicalizations of Finite Projective Planes

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SUMRY 2016
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Projective Plane

Definition

The **Projective Plane** $\mathbb{P}^2_K$ over a field K is the set

$$\left(K^3 \backslash \{\vec{0}\}\right) \Big/ \{\vec{x} \sim \lambda \vec{x}\}$$

where $\lambda \in K^*$.

Points in $\mathbb{P}^2_K$ are denoted by $(x_0 : x_1 : x_2)$.

Fano Arrangement

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Example of a Line:

$$\begin{aligned} \ell &= \{(x_0 : x_1 : x_2) : x_0 + x_2 = 0\} \\ &= \{(0:1:0), (1:0:1), (1:1:1)\}. \end{aligned}$$

Fano Arrangement

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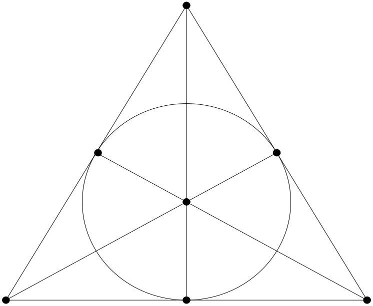

Figure 1: Fano Arrangement

Laurent Series

$K := \mathbb{F}_2(\!(t)\!)$ - Laurent series in the variable t.

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Example:

$$a_0 = t^{-2} + 1 + t^5 + \cdots$$

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The **t-adic valuation** is a map $\nu : K^* \to \mathbb{Z}$.

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Example:

$$a_0 = t^{-2} + 1 + t^5 + \cdots \Rightarrow \nu(a_0) = -2$$

Lines in $\mathbb{P}^2_K$ to their "shadows" in $\mathbb{R}^2$

Given a point $p = (a : b : c) \in \mathbb{P}^2_K$ the **tropicalization** of p is given by $\text{Trop}(p) = \text{Trop}((a : b : c)) = [\nu(a) : \nu(b) : \nu(c)]$.

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Definition

Trop(p) is a point in the **tropical projective plane**,

$$\mathbb{P}^2_{\mathrm{Trop}} = (\mathbb{R} \cup \{\infty\})^3 \setminus \{\vec{\infty}\} \Big/ \{\vec{z} + \lambda \mathbb{1} \sim \vec{z}\}.$$

Example:

$$[2 : 3 : 2] = [0 : 1 : 0]$$

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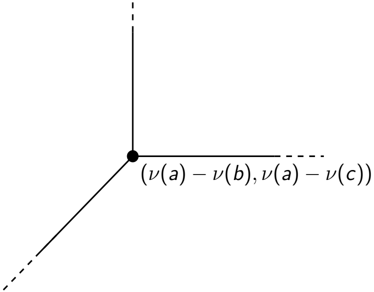
Note that $[a + \lambda : b + \lambda : c + \lambda]$ is plotted at
$((a + \lambda) - (b + \lambda), (a + \lambda) - (c + \lambda)) = (a - b, a - c)$
So this map is well defined.

Lines in $\mathbb{P}^2_K$ to their "shadows" in $\mathbb{R}^2$

Trop($\{ax_0 + bx_1 + cx_2 = 0\}$) is the **standard tropical line**,



whose vertex is the point $[\nu(a) : \nu(b) : \nu(c)]$ which we identify with the point $(\nu(a) - \nu(b), \nu(a) - \nu(c))$ in $\mathbb{R}^2$.

Classification

Goal

Let $\mathcal{A}$ be the Fano Arrangement. Given a matrix $g \in \mathrm{GL}(3, K)$, we want to give a combinatorial classification for the tropical line arrangements $\mathrm{Trop}(g \cdot \mathcal{A})$.

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Conditions from the Reduction

If $g \cdot \ell = \{ax_0 + bx_1 + cx_2 = 0\}$, and $\bar{g} \cdot \ell = \{x_0 = 0\}$, then $\nu(a) = 0$ and $\nu(b), \nu(c) > 0$.

In general, a variable appears in the equation for $\bar{g} \cdot \ell$ if and only if the valuation of its coefficient in the equation for $g \cdot \ell$ is zero.

Example

The **vertex** of $\mathrm{Trop}(ax_0 + bx_1 + cx_2 = 0)$ is $[\nu(a) : \nu(b) : \nu(c)]$.

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An Observation

Proposition

Given a column $\vec{v}$ of a matrix $g \in \mathrm{GL}(3, K)$

1. There are exactly three valuations attainable as $\mathbb{F}_2$-linear combinations of the entries of $\vec{v}$.
2. There is exactly one $\mathbb{F}_2$-linear combination that produces the largest of those valuations.

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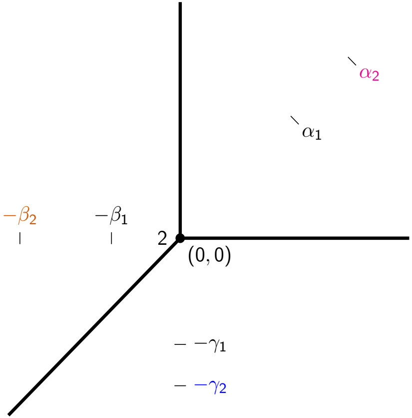
3. The largest valuation for each column (α_2, β_2, and γ_2) appears in exactly one vertex.

Plotting the Tropicalization

Using the vertices of the tropical lines obtained using $\bar{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$:



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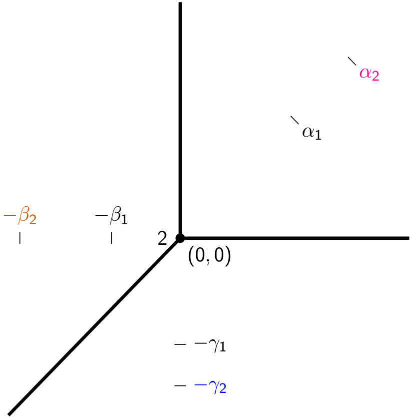

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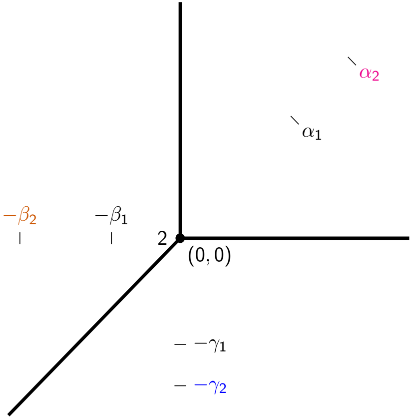

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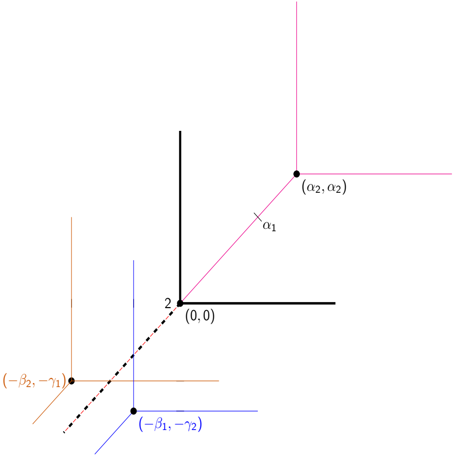

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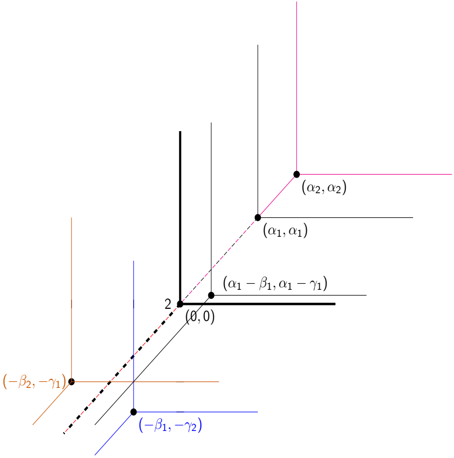

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Main Result

Theorem (B. M. R. W. '16)

A tropical line arrangement $\mathcal{T}$ of seven lines is a tropicalization of $g \cdot \mathcal{A}$ if and only if:

1. There exist 9 integers forming **valuation sequences**
$$0 = \alpha_0 < \alpha_1 < \alpha_2$$
$$0 = \beta_0 < \beta_1 < \beta_2$$
$$0 = \gamma_0 < \gamma_1 < \gamma_2$$
such that the vertices of the lines in $\mathcal{T}$ may be written as $[\alpha_i : \beta_j : \gamma_k]$.
2. The zeros occur in one of four particular configurations.
3. The maximums $\alpha_2, \beta_2,$ and γ_2 each occur in exactly one vertex.

Further Results

Define $\mathcal{A}_q$ as the set of the $q^2 + q + 1$ lines in $\mathbb{P}^2_{\mathbb{F}_q}$, where q is a prime power.

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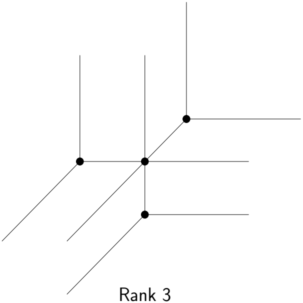
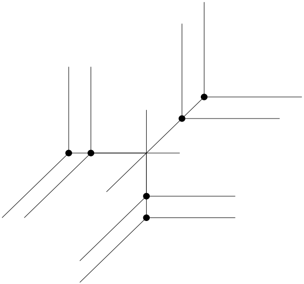
Theorem (Extension to $\mathbb{F}_q$)

A tropical line arrangement $\mathcal{T}_q$ of $q^2 + q + 1$ lines is a tropicalization of $g \cdot \mathcal{A}_q$ if and only if the previous three conditions hold.

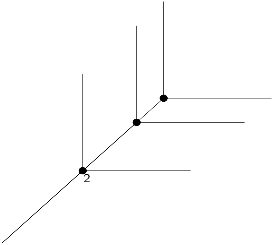
Center Behaviors



Rank 3



Rank 2, $e_i \notin \operatorname{Im} \bar{g}^\top$



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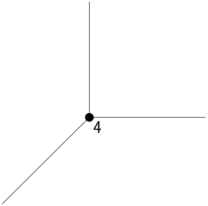

Rank 1

Thank You!