Sequence non-squashing partitions

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July 25, 2014

Integer partitions

Definition

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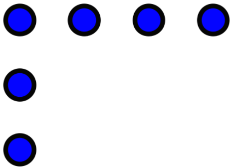
- Denote the *partition function* by

$$p(n) := \#\{\text{partitions of } n\}.$$

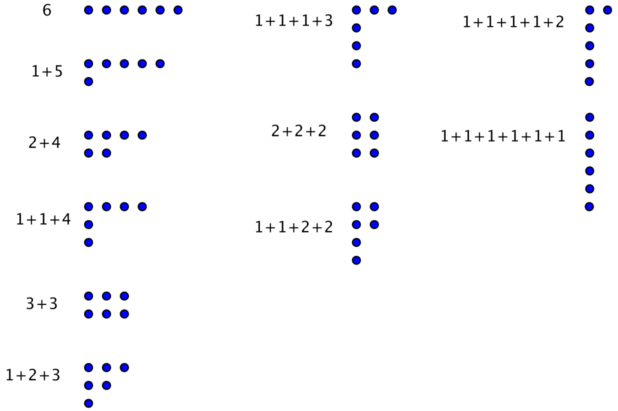
Ferrers diagram

$$1 + 1 + 4 = 6$$



Example

$$p(6) = 11$$



List of integer partitions

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12	77
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15	176
16	231
17	297
18	385
19	490
20	627
21	792
22	1002
23	1255
24	1575
25	1958
26	2436
27	3010

n	$p(n)$
28	3718
29	4565
30	5604
31	6842
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105558	124754	147273	173525	204226
239943	281589	329931	386155	451276
526823	614154	715220	831820	966467
1121505	1300156	1505499	1741630	2012558
2323520	2679689	3087735	3554345	4087968
4697205	5392783	6185689	7089500	8118264

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Ramanujan congruences

Theorem (Ramanujan, 1919)

For all integers $n \geq 0$,

- $p(5n+4) \equiv 0 \bmod 5$,

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For all integers $n \geq 0$,

- $p(5n+4) \equiv 0 \bmod 5$,
- $p(7n+5) \equiv 0 \bmod 7$,
- $p(11n+6) \equiv 0 \bmod 11$.

Growth of integer partitions

n	$p(n)$
1	1
10	4.2×10^{1}
100	1.9×10^{8}
1000	2.4×10^{31}
10000	3.6×10^{106}
100000	2.7×10^{346}
1000000	1.5×10^{1107}
10000000	9.2×10^{3514}
100000000	1.8×10^{11131}
1000000000	1.6×10^{35218}
10000000000	1.1×10^{111390}

Hardy-Ramanujan asymptotics

Theorem (Hardy-Ramanujan, 1918)

As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

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Remark

We say $f(n) \sim g(n)$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Example

$$\text{Let } a(n) := \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

n	$p(n)$	$a(n)$	$\frac{p(n)}{a(n)}$
1	1.000×10^{0}	1.876×10^{0}	0.5328
10	4.200×10^{1}	4.810×10^{1}	0.8731
10^2	1.905×10^{8}	1.992×10^{8}	0.9562
10^3	2.406×10^{31}	2.440×10^{31}	0.9860
10^4	3.616×10^{106}	3.632×10^{106}	0.9955
10^5	2.749×10^{346}	2.753×10^{346}	0.9985
10^6	1.471×10^{1107}	1.472×10^{1107}	0.9995
10^7	9.202×10^{3514}	9.204×10^{3514}	0.9998
10^8	1.760×10^{11131}	1.760×10^{11131}	0.9999

m-ary partitions

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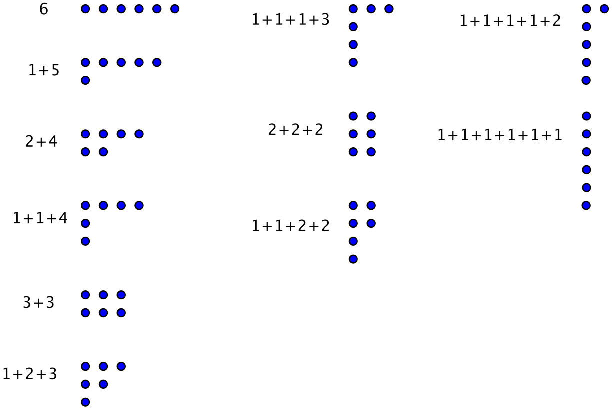
- Denote the *m-ary partition function* by

$$\beta_m(n) := \#\{m\text{-ary partitions of } n\}.$$

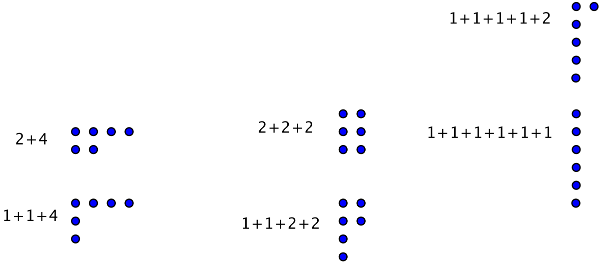
Example

$$p(6) = 11$$



Example

$$\beta_2(6) = 6$$



Churchhouse conjectures

Conjecture (Churchhouse, 1969)

For all integers $k \geq 1$ and odd n,

$$\beta_2(2^{2k+2}n) \equiv \beta_2(2^{2k}n) \bmod 2^{3k+2}$$

$$\beta_2(2^{2k+1}n) \equiv \beta_2(2^{2k-1}n) \bmod 2^{3k}$$

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- Proved by Rodseth (1970)
- Extended by Andrews (1971) and Gupta (1972)

m-ary partition function asymptotics

Theorem (Mahler, 1940)

For all integers $m \geq 2$, as $n \to \infty$,

$$\ln \beta_m(n) \sim \frac{(\ln n)^2}{2 \ln m}.$$

m-non-squashing partitions

Definition (Hirschhorn-Sellers, 2004)

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- Denote the *m-non-squashing partition function* by

$$\alpha_m(n) := \#\{m\text{-non-squashing partitions of } n\}.$$

The non-squashing condition

Example

The 2-non-squashing condition can be written out as:

$$\begin{aligned} p_1 &\leq p_2, \\ p_1 + p_2 &\leq p_3, \\ p_1 + p_2 + p_3 &\leq p_4. \\ &\vdots \end{aligned}$$

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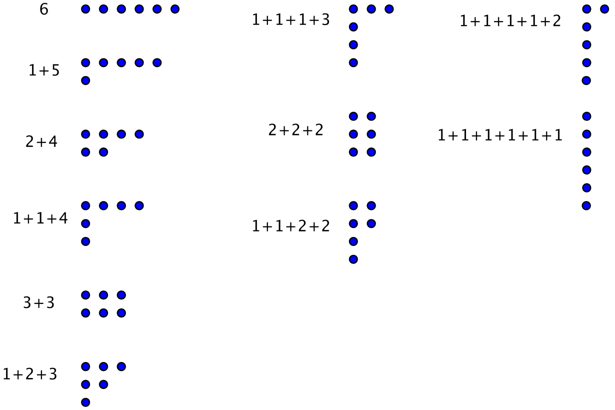

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$$\alpha_2(6) = 6$$

6

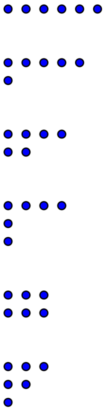
1+5

2+4

1+1+4

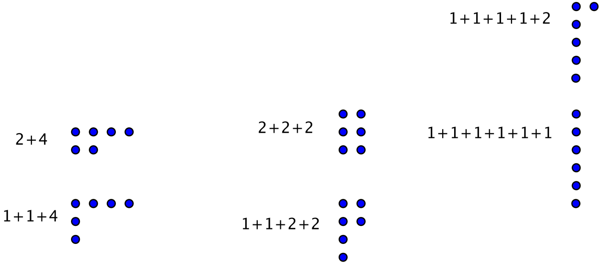
3+3

1+2+3



Example

$$\alpha_2(6) = 6 = \beta_2(6)$$



A surprising connection!

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Theorem (Hirschhorn-Sellers, 2004)

For all integers $m \geq 2$ and $n \geq 0$,

$$\alpha_m(n) = \beta_m(n).$$

Other connections to m-non-squashing partitions

Other connections to *m*-non-squashing partitions

- Recursively self-conjugating partitions (Keith, 2011)
- Double base number system (Dmitrov-Imbert-Mishra, 2008)
- Box stacking problem (Sloane-Sellers, 2005)

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- Recursively self-conjugating partitions (Keith, 2011)
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- Box stacking problem (Sloane-Sellers, 2005)
- Partitions with factorial parts (?) (Andrews-Sellers, Jovovic, 2007)

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Sequence non-squashing partitions

Definition (Homma-Ryu-Tong)

- Let $M = \{m_j\}_{j=0}^{\infty}$ with integers $m_j \geq 2$ be a sequence. A *M-sequence non-squashing partition* of an integer $n \geq 0$ is a set of positive integers $\{p_1, \ldots, p_k\}$ such that

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- *If* $n = p_1 + p_2$, *then*
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- *If* $n = p_1 + p_2 + p_3$, *then*
$$(m_0 - 1)(p_1 + p_2) \leq p_3,$$
$$(m_1 - 1)p_1 \leq p_2.$$

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- *If* $n = p_1 + p_2 + p_3$, *then*
$$\begin{aligned}(m_0 - 1)(p_1 + p_2) &\leq p_3,\\ (m_1 - 1)p_1 &\leq p_2.\end{aligned}$$

- *If* $n = p_1 + p_2 + p_3 + p_4$, *then*
$$\begin{aligned}(m_0 - 1)(p_1 + p_2 + p_3) &\leq p_4,\\ (m_1 - 1)(p_1 + p_2) &\leq p_3,\\ (m_2 - 1)p_1 &\leq p_2.\end{aligned}$$

Returning to m-non-squashing

Remark

Let $M = \{m, m, m, \ldots\}$. Then,

$$\alpha_M(n) = \alpha_m(n) = \beta_m(n).$$

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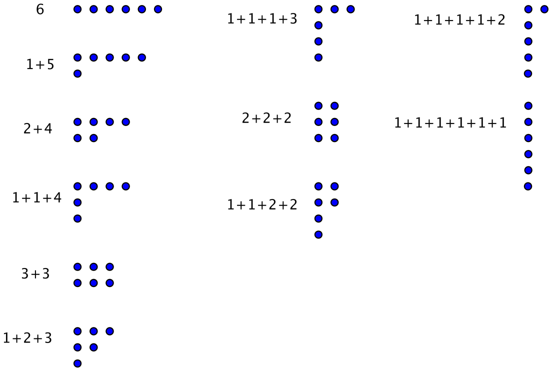
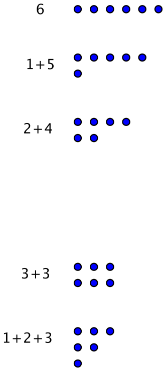


Example

$$\alpha_M(6) = 5 \text{ for } M = M_f = \{2, 3, 4, \ldots\}$$



Relationship to factorial partitions

Proposition (Homma-Ryu-Tong)

Let $M_f = \{2, 3, 4, \ldots\}$. *Then,*

$$\alpha_{M_f}(n) = f(n),$$

where $f(n)$ *is the number of factorial partitions of* n.

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  - each $p_j = \ell_j!$ for some integer $\ell_j \geq 1$.
- Denote the *factorial partition function* by

$$f(n) := \#\{\text{factorial partitions of } n\}.$$

Motivating questions

- What can be said if "m" changes with the number of parts?
  - *Can we discover and recover congruences?*
  - Can we discover and recover asymptotics?

Rodseth-Sellers congruences

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Theorem (Rodseth-Sellers, 2000)

For all integers $m \geq 2, n, r \geq 1$,

$$\beta_m(m^{r+1}n + \sigma_{m,r}) \equiv 0 \bmod \frac{m^r}{c_r},$$

where $c_r = 2^{r-1}$ if m is even and 1 if m is odd.

Sequence non-squashing congruences

Theorem (Homma-Ryu-Tong)

Let $P_j = \prod_{\text{prime } p \leq j} p$ *and* $\mu_j = \frac{m_j}{\gcd(m_j, P_j)}$. *Then, for all integers* $n, r \geq 1$,

$$\alpha_M(nm_0m_1 \ldots m_r + \sigma_{M,r}) \equiv 0 \bmod \mu_1\mu_2 \ldots \mu_r.$$

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Remark

If $M = \{m, m, m, \dots\}$, then this theorem weakly recovers the Rodseth-Sellers congruence.

Factorial partition congruences

Corollary (Homma-Ryu-Tong)

For all integers $r \geq 3, n \geq 1,$ *let* $D_r = \prod_{\text{prime } p \leq r-2} p^{\left\lfloor \frac{r-2}{p} \right\rfloor}$. *Then,*

$$f(r!n + \sigma_r) \equiv 0 \bmod \frac{r!}{D_r}.$$

Examples

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18	22
19	22
20	26
21	26
22	30
23	30
24	36
25	36
26	42
27	42

n	$f(n)$
28	48
29	48
30	56
31	56
32	64
33	64
34	72
35	72
36	82
37	82
38	92
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40	102
41	102

Examples

$$f(3!n-2) = f(3!n-1) \equiv 0 \bmod 3$$

1	1	2	2	3	3
5	5	7	7	9	9
12	12	15	15	18	18
22	22	26	26	30	30
36	36	42	42	48	48
56	56	64	64	72	72
82	82	92	92	102	102
114	114	126	126	138	138
153	153	168	168	183	183
201	201	219	219	237	237
258	258	279	279	300	300
324	324	348	348	372	372
400	400	428	428	456	456
488	488	520	520	552	552

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Motivating questions

- What can be said if "m" changes with the number of parts?
  - Can we discover and recover congruences?
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Sequence non-squashing asymptotics

Theorem (Homma-Ryu-Tong)

For all integers $n \geq m_0$,

$$\frac{n^N}{N! \prod_{k=0}^{N-1} m_k^{N-k}} \leq \alpha_M(n) < \frac{2^N n^N}{\prod_{k=0}^{N-1} m_k^{N-k}},$$

where N is the unique integer such that:

$$m_0 m_1 \cdots m_{N-1} \leq n < m_0 m_1 \cdots m_N.$$

Application to the m-ary partition function

Corollary (Homma-Ryu-Tong)

For all integers $m \geq 2$, as $n \to \infty$, we recover Mahler's asymptotic

$$\ln \beta_m(n) \sim \frac{(\ln n)^2}{2 \ln m}.$$

Factorial partition function asymptotics

Theorem (Homma-Ryu-Tong)

As $n \to \infty$,

$$\ln f(n) \sim \frac{(\ln n)^2}{2 \ln \ln n}.$$

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Sequence non-squashing recurrence relations

Proposition (Homma-Ryu-Tong)

For all integers $n \geq 1$,

$$\alpha_M(n) = \begin{cases} \alpha_M(n-1) & \text{if } m_0 \nmid n \\ \alpha_M(n-1) + \alpha_{\overline{M}}\left(\dfrac{n}{m_0}\right) & \text{if } m_0 \mid n, \end{cases}$$

where $\overline{M} = \{m_j\}_{j=1}^{\infty}$.

Summation formula

Proposition (Homma-Ryu-Tong)

For all integers $n \geq m_0$,

$$\alpha_M(n) = \sum_{k_1=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \sum_{k_2=0}^{\left\lfloor \frac{k_1}{m_1} \right\rfloor} \cdots \sum_{k_N=0}^{\left\lfloor \frac{k_{N-1}}{m_{N-1}} \right\rfloor} 1,$$

where N *is the unique integer such that*

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Future directions

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- Extend our results to other sequences M.

Acknowledgements

- Professor Amanda Folsom, Yale University
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- Audience!

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