Body-and-CAD Rigidity & Projective Geometry

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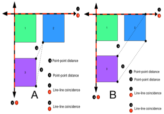
Motivation

We work with body-and-CAD frameworks, which are a mathematical model for structures that arise in constraint-based computer-aided design software. We analyze such frameworks in order to detect special positions with the goal of automated detection.

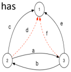
Body and CAD frameworks



Framework A admits no internal degrees of freedom, called a *generic position*. In framework B, if we tie down body 3, we can still move 1 and 2. We call this a *special position* of the first framework.

The Primitive CAD Graph

A body-and-CAD framework with *n* bodies can be represented as a graph, with the bodies as vertices and the constraints, which eliminate internal degrees of freedom, as edges.[1] We represent the tie-down vertex by directing all incident edges towards it, and then directing the remaining edges such that each vertex has equal outward degree.



Pictured: The graph of frameworks A and B, with 1 tied down.

The Rigidity Matrix

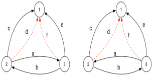
Rows are constraints that correspond to graph edges. In order to represent tying a body down, we append the identity matrix to the columns corresponding to the rows. Matrices with rank less than $3n$-3 represent flexible frameworks.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -2 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Pictured above: the rigidity matrix for framework A.

The Pure Condition as a Bracket Polynomial

Partition the edges and hence the rows of the rigidity matrix based on the out degree of each unpinned vertex. We can associate a product of brackets to each partition, the sum over each product of brackets for each distinct directed graph of the tied-down vertex yields the pure condition of the graph, which is the determinant of the rigidity matrix.[2]

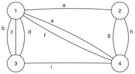


The two distinct directed graphs for pinning 1

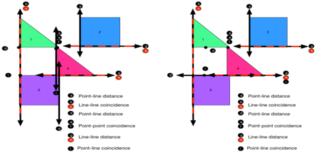
Thus the pure condition for pinning vertex 1 on the above graphs is:
[acd][bef]-[bcd][aef]

An Inductive Procedure for Constructing Rigid Graphs

We use an inductive procedure for constructing rigid graphs (Henneberg moves) and analyze their special positions. Their pure condition is a product of brackets. A bracket will vanish if the points dual to the lines represented by the edges in the real projective plane are collinear. [2]



Pure condition of graph to the right by pinning 1:
[agh][bcd][efi]



Pictured above: 4-body framework generated through Henneberg I moves, with a special position found by analyzing the pure condition.

Applications

Using the pure condition to detect special positions of body-and-CAD frameworks, we may give users information about CAD structures while retaining the intuition of the geometric constraint system.

References & Acknowledgements

[1] White and Whiteley, *The Algebraic Geometry of Motions of Bar-and-Body Frameworks*. Siam J. Alg. Disc. Meth. Vol. 8, No. 1, January 1987.
[2] Farre, Kleinschmidt, Sidman, St. John, Stark, and Theran, *Detecting Dependencies in Geometric Constraint Systems*. Proceedings of the 10th International Workshop on Automated Deduction in Geometry (ADG 2014), 9-11 July 2014. University of Coimbra, Portugal. pp. 147-166.