Simplicial Complexes and Effective Divisors of $\overline{M}_{0,n}$

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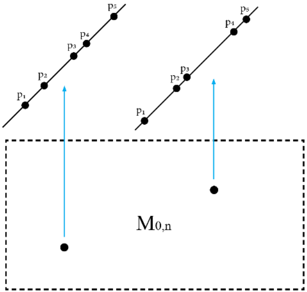

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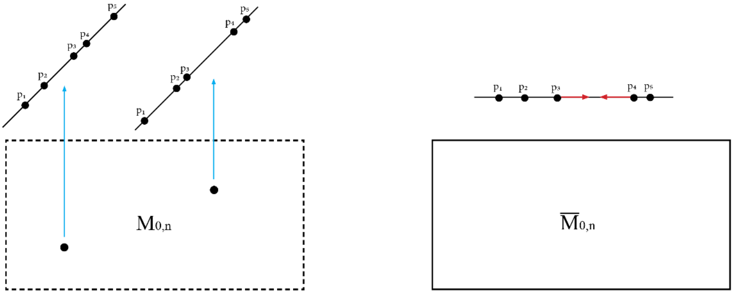

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- The set of effective divisors is a non-finitely generated monoid contained in this group.

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- This makes the problem of determining which divisors are equivalent to effective divisors challenging.

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2. Find minimal generators for the monoid of effective divisors on $\overline{M}_{0,7}$

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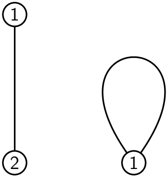

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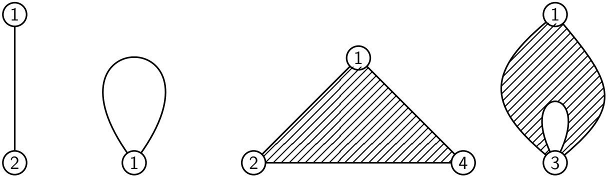

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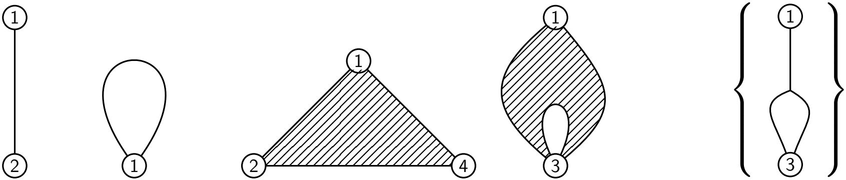

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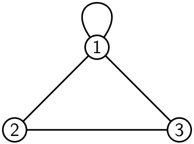

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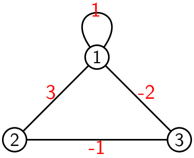

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A weighting on a d-complex Δ is *balanced* if, for each multiset S of cardinality d such that each element of S is in A,

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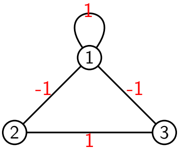

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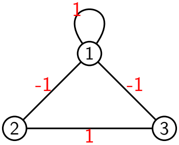

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Every d-complex Δ on $n-1$ vertices corresponds to a divisor class D_Δ in $\overline{M}_{0,n}$ defined as follows:

$$D_\Delta := (d+1)H - \sum_I \left(d+1 - \max_{\sigma\in\Delta}\left\{\sum_{i\in I} mult_i(\sigma)\right\}\right)E_I$$

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If D is effective, and $D - \sum E_I$ is not effective for any $\sum E_I$, then there exists a properly balanceable complex Δ such that $D_\Delta = D$.

Known Families of Minimal Effective Divisors on $\overline{M}_{0,7}$

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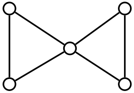
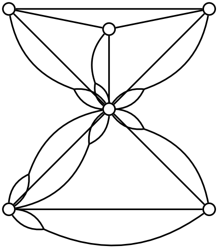
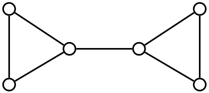
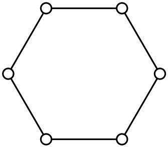
$M_1 =$  , $M_2 =$  , $M_3 =$ 

(Boundary) (Keel-Vermeire) (Opie)

$M_4 =$  , $M_5 =$ 

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Theorem (Effective Divisor Criterion)

A divisor D is effective if and only if $\Delta(D)$ is balanceable.

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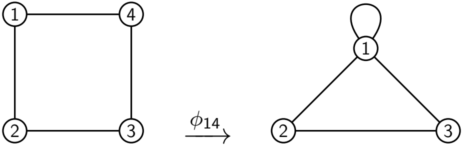

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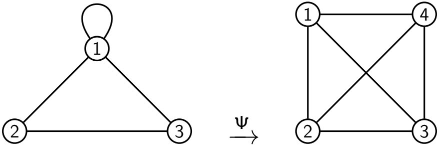

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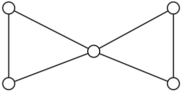
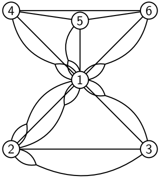
$M_1 =$  , $M_2 =$  , $M_3 =$ 

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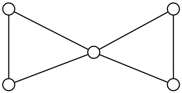
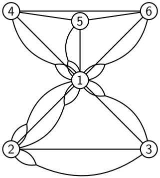
$M_1 =$  , $M_2 =$  , $M_3 =$ 

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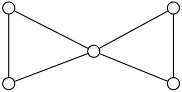
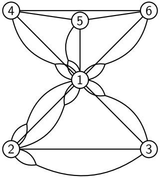
$M_1 =$  , $M_2 =$  , $M_3 =$ 

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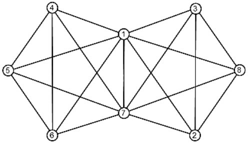

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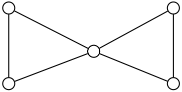
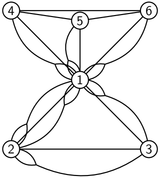
$M_1 =$  , $M_2 =$  , $M_3 =$ 

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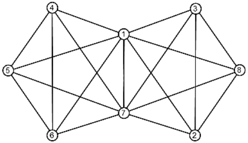
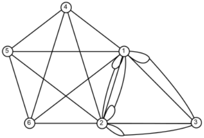


Question: Is there another way of identifying vertices on this complex to get a complex corresponding to a minimal effective divisor?

A New Minimal Effective Divisor on $\overline{M}_{0,7}$



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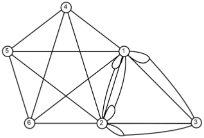

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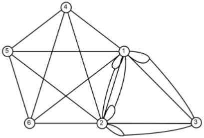

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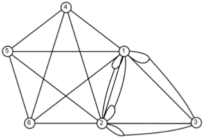

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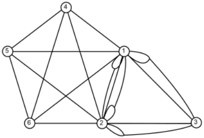
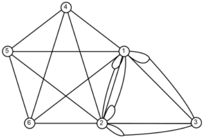
Corollary

D_Δ is a minimal effective divisor.

Thank You!

$M_6 =$ 

Thank You!

$M_6 =$ 

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Thanks for listening!

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