Simplicial Complexes and Effective Divisors of $\overline{M}_{0,n}$

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$\overline{M}_{0,n}$: Our Ambient Space
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\( M_{0,n} : \) The moduli space parameterizing configurations of \( n \) distinct points on \( \mathbb{CP}^1 \) under the action of \( PGL_2(\mathbb{C}) \).

\( \overline{M}_{0,n} : \) A space parameterizing these configurations and their limits.
Definition

A divisor $D$ is a formal linear sum of codimension-1 subvarieties $D_i$:

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- The set of divisors is a non-finitely generated abelian group.
- The set of effective divisors is a non-finitely generated monoid contained in this group.
There exists an equivalence relation between divisors on $\overline{M}_{0,n}$, determined by intersection of divisors with curves.
Divisors on $\overline{M}_{0,7}$

- There exists an equivalence relation between divisors on $\overline{M}_{0,n}$, determined by intersection of divisors with curves.

- Under this equivalence relation the group of divisors on $\overline{M}_{0,7}$ is isomorphic to $\mathbb{Z}^{42}$, with basis $H, E_i, E_{ij}, E_{ijk}$ where $i, j, k \in \{1, \ldots, 6\}$ are distinct.
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Under this equivalence relation the group of divisors on \( \overline{M}_{0,7} \) is isomorphic to \( \mathbb{Z}^{42} \), with basis \( H, E_i, E_{ij}, E_{ijk} \) where \( i, j, k \in \{1, \ldots, 6\} \) are distinct.

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Under this equivalence relation there exist divisors with negative coefficients that are equivalent to effective divisors.

This makes the problem of determining which divisors are equivalent to effective divisors challenging.
Understand effective divisors on $\overline{M}_{0,n}$:
Project Goal

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1. Classify effective divisors (necessary and sufficient conditions)
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1. Classify effective divisors (necessary and sufficient conditions)

2. Find minimal generators for the monoid of effective divisors on $\overline{M}_{0,7}$
A \textit{d-simplex} \( \sigma \) on a set \( A \) is a multiset of elements in \( A \) with cardinality \( d + 1 \).
**Definition**

A *d-simplex* $\sigma$ on a set $A$ is a multiset of elements in $A$ with cardinality $d + 1$.

**Example**: Let $A = \{1, 2, 3, 4\}$. Then $\{1, 2\}$ and $\{1, 1\}$ are 1-simplices and $\{1, 2, 4\}$ and $\{1, 3, 3\}$ are 2-simplices.
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Simplicial Complexes

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A \emph{d-complex} \( \Delta \) on a set \( A \) is a set of \( d \)-simplices \( \sigma_i \) on \( A \):
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\Delta := \{ \sigma_1, \ldots, \sigma_r \}.
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**Example:** \( \Delta = \{ \{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \} \) is a 1-complex.
Weighting

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A \textit{d-complex} $\Delta$ on a set $A$ is a set of $d$-simplices $\sigma_i$ on $A$:

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Definition

A \textit{weighting} on a \textit{d-complex} $\Delta = \{\sigma_1, \ldots, \sigma_r\}$ is an assignment of an integer $w_i$ to each simplex $\sigma_i$. 
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\textbf{Example:} \( \Delta = \{\{1,1\} : 1, \{1,2\} : 3, \{1,3\} : -2, \{2,3\} : -1\} \).
Balancing

Definition

A weighting on a $d$-complex $\Delta$ is *balanced* if, for each multiset $S$ of cardinality $d$ such that each element of $S$ is in $A$,

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\sum_{\sigma_i \supseteq S} w_i \cdot \text{mult}(S \subseteq \sigma_i) = 0.
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A $d$-complex $\Delta$ is *balanceable* if there exists a balanced weighting on $\Delta$. 
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A $d$-complex $\Delta$ is balanceable if there exists a balanced weighting on $\Delta$.

Example: $\Delta = \{\{1, 1\} : 1, \{1, 2\} : -1, \{1, 3\} : -1, \{2, 3\} : 1\}$. 

![Diagram of a 1-complex with weights 1, -1, -1, 1 at vertices 1, 2, 3, respectively.]
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Definition

A balanced weighting is called proper if $w_i \neq 0$ for all $i$. 

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Special Balancings

Definition
A weighting on a $d$-complex $\Delta$ is balanced if, for each multiset $S$ of cardinality $d$ such that each element of $S$ is in $A$,

$$\sum_{\sigma_i \supseteq S} w_i \cdot \text{mult}(S \subseteq \sigma_i) = 0.$$ 

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A balanced weighting is called proper if $w_i \neq 0$ for all $i$.

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A $d$-complex is simply balanceable if there exists a unique properly balanced weighting up to scaling.
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A $d$-complex is \textit{simply balanceable} if there exists a unique properly balanced weighting up to scaling.

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Definition

A $d$-complex is *simply balanceable* if there exists a unique properly balanced weighting up to scaling.

**Example:** $\Delta = \{\{1, 1\} : 1, \{1, 2\} : -1, \{1, 3\} : -1, \{2, 3\} : 1\}$. 

![Diagram of a complex with weights 1 and -1 on edges]
Every $d$-complex $\Delta$ on $n - 1$ vertices corresponds to a divisor class $D_\Delta$ in $\overline{M}_{0,n}$ defined as follows:

$$D_\Delta := (d + 1)H - \sum_{l} \left( d + 1 - \max_{\sigma \in \Delta} \left\{ \sum_{i \in l} \text{mult}_i(\sigma) \right\} \right) E_l$$

where $1 \leq |l| \leq n - 4$ and $l \subset \{1, \ldots, n - 1\}$. 
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**Theorem**

*If a complex $\Delta$ is properly balanceable, then $D_\Delta$ is effective.*
Every $d$-complex $\Delta$ on $n - 1$ vertices corresponds to a divisor class $D_\Delta$ in $\overline{M}_{0,n}$ defined as follows:

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where $1 \leq |I| \leq n - 4$ and $I \subseteq \{1, \ldots, n - 1\}$.

**Theorem**

*If a complex $\Delta$ is properly balanceable, then $D_\Delta$ is effective.*

**Theorem**

*If $D$ is effective, and $D - \sum E_I$ is not effective for any $\sum E_I$, then there exists a properly balanceable complex $\Delta$ such that $D_\Delta = D$.***
Known Families of Minimal Effective Divisors on $\overline{M}_{0,7}$

- Exceptional Divisors: $E_i, E_{ij}, E_{ijk}, i, j, k \in \{1, \ldots, 6\}$
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- Non-exceptional ("Horizontal") Divisors:

$$M_1 = \circ \circ, \quad M_2 = \bigcirc \bigcirc \bigcirc, \quad M_3 =$$
(Boundary) (Keel-Vermeire) (Opie)

$$M_4 = \bigcirc \bigcirc \bigcirc, \quad M_5 =$$
(DGJ) (Castravet-Tevelev)
Our Results

Definition

A complex $\Delta$ is said to be complete when $D_{\Delta \cup \{\sigma\}} = D_\Delta$ only if $\sigma \in \Delta$. 
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- There is a unique complete complex affiliated with any divisor $D$, denoted by $\Delta(D)$. 
Our Results

Definition
A complex $\Delta$ is said to be complete when $D_{\Delta \cup \{\sigma\}} = D_{\Delta}$ only if $\sigma \in \Delta$.

- There is a unique complete complex affiliated with any divisor $D$, denoted by $\Delta(D)$.

Theorem (Effective Divisor Criterion)
A divisor $D$ is effective if and only if $\Delta(D)$ is balanceable.
Vertex Identification
Let $\phi_{ij} : \Delta_n \to \Delta_{n-1}$ be the map that sends complexes on $n$ indices to complexes on $n - 1$ by replacing the $j$-th index with $i$. 
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Example:

$$\phi_{14} (\{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}) = \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$$
**Vertex Identification**

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Remark

There exists a canonical reverse $\Psi$ of a series of vertex identifications that generates a complex with no degeneracy.
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There exists a canonical reverse $\Psi$ of a series of vertex identifications that generates a complex with no degeneracy.

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A Pattern

\[ M_1 = \circ \circ , \quad M_2 = , \quad M_3 = \]
A Pattern

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M_1 = \circ \circ \circ \circ \circ,
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M_3 = \circ \circ \circ \circ \circ
\]

Note that \( \Psi(M_3) \) gives the complex:
A Pattern

\[ M_1 = \circ \circ , \quad M_2 = , \quad M_3 = \frac{1}{2} \frac{2}{3} \frac{4}{5} \frac{6}{} \]

Note that \( \Psi(M_3) \) gives the complex:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
4 & 5 & 6 & 1 \\
2 & 3 & 1 & 5 \\
6 & 1 & 3 & 2
\end{array}
\]
Note that $\Psi(M_3)$ gives the complex:

Question: Is there another way of identifying vertices on this complex to get a complex corresponding to a minimal effective divisor?
A New Minimal Effective Divisor on $\overline{M}_{0,7}$

With corresponding divisor:

$$3H - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - E_{14} - E_{15} - E_{16} - E_{24} - E_{25} - 2E_{34} - 2E_{35} - 2E_{36} - E_{45} - E_{46} - E_{56} - E_{345} - E_{346} - E_{356}$$
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$\Delta$ is complete, and $D_\Delta$ cannot be written as $\sum_i a_i M_i$
A New Minimal Effective Divisor on $\overline{M}_{0,7}$

With corresponding divisor:

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$\Delta$ is complete, and $D_{\Delta}$ cannot be written as $\sum_i a_i M_i$.

**Theorem (Strict Effectiveness)**

*If $\Delta$ is a simply balanceable and complete complex, then $D_{\Delta}$ breaks as a strict sum of non-exceptional minimal effective divisors.*
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\( \Delta \) is complete, and \( D_\Delta \) cannot be written as \( \sum_i a_i M_i \)

**Theorem (Strict Effectiveness)**

*If \( \Delta \) is a simply balanceable and complete complex, then \( D_\Delta \) breaks as a strict sum of non-exceptional minimal effective divisors.*

**Corollary**

*\( D_\Delta \) is a minimal effective divisor.*
Thank You!

\[ M_6 = \]
We’d like to thank the MAA for hosting MathFest! Additionally, we’d like to thank José González for advising us, as well as Michael Magee and Sam Payne for organizing SUMRY.

Thanks for listening!
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