Counting (a, b)-Core Partitions from Numerical Semigroups

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Motivating Question

For coprime integers a and b, how many (a, b)-core partitions come from numerical semigroups? How does this compare to the total number of (a, b)-cores?

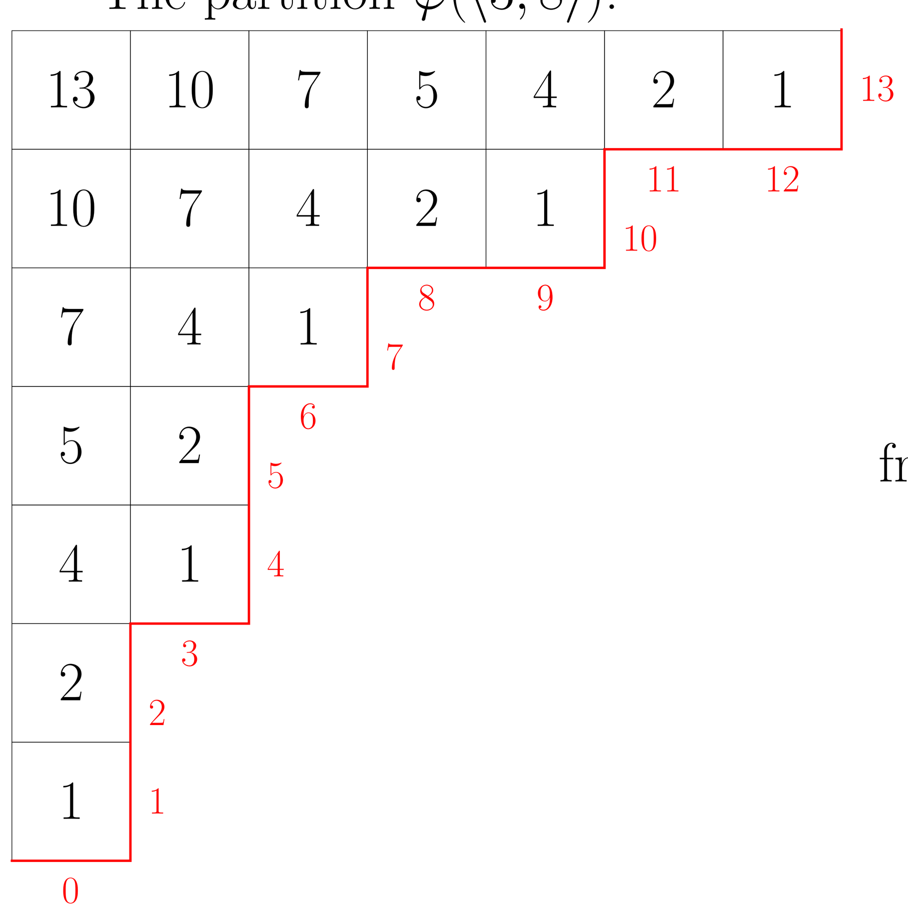
Definitions and Examples

A **numerical semigroup** S is a subset of $\mathbb{N}$ that is closed under addition, contains 0, and has a finite complement.

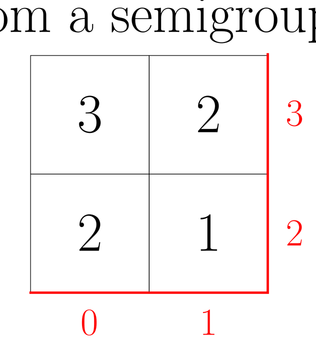
Ex: $S = \langle 3, 8\rangle = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, \ldots\}$.
$\mathbb{N} \setminus S = \{1, 2, 4, 5, 7, 10, 13\}$.

Given a numerical semigroup S, one can form an integer partition $\varphi(S)$ by constructing a Young diagram via a simple algorithm. The partition $\varphi(S)$ is the number of boxes in each row of the Young diagram. In this example, $\varphi(S) = (7, 5, 3, 2, 2, 1, 1)$. Note that not all partitions come from semigroups—for instance $(2, 2)$.

The partition $\varphi(\langle 3, 8\rangle)$:



A partition not from a semigroup:



Given a Young diagram, each box has a **hook length**, the number of boxes below and to the right of the box, plus one for the box itself. The **hook set** of a partition is the set of hook lengths of its associated Young diagram. In the examples the hook lengths have been filled in with black.

A partition is called an ***a*–core** if none of its associated hook lengths are divisible by a. Similarly, a partition is called an **(a,b)–core** if it is both an a-core and a b-core. In our example, $\varphi(S)$ is a $(3, 8)$-core.

Known Results

We would like to compare the number of (a, b)-cores from semigroups with the total number of (a, b)-core partitions, which is given by:

Theorem (Anderson, [1])

If $\gcd(a, b) = 1$, the total number of (a, b)-core partitions is
$$\frac{1}{a+b}\binom{a+b}{a}$$

Apéry Tuples and Polytopes

- We fix $S = \langle a, b\rangle$. The set of semigroups containing S are the **oversemigroups** of S, denoted by $O(S)$.
- A partition $\varphi(T)$ is an (a, b)-core if and only if $a, b \in T$.

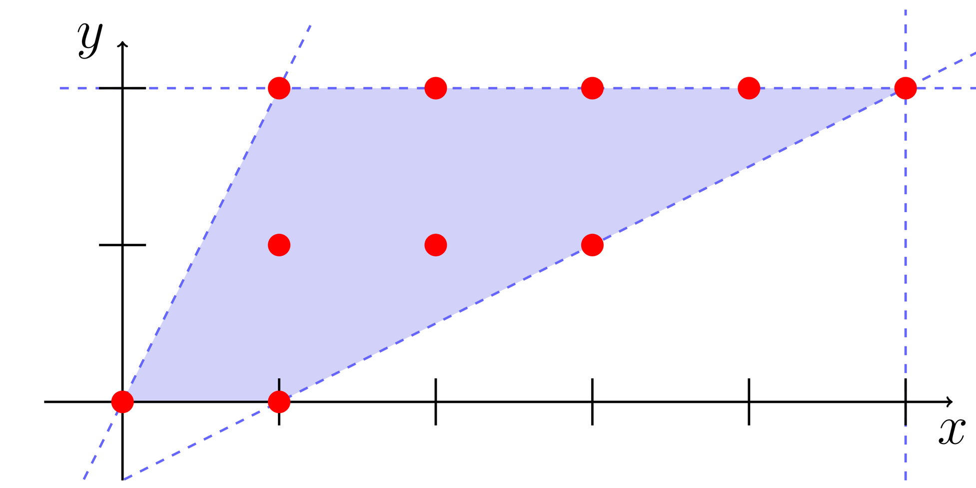
Therefore, $|O(S)|$ equals the number of (a, b)-cores from semigroups.

The **Apéry tuple** of S with respect to some $n \in S$, denoted $\mathrm{Ap}'(S, n)$, is the tuple of k_i such that $n \cdot k_i + i$ is the smallest element of S congruent to $i \pmod n$ for $i \in \{1, \ldots, n-1\}$.

Assume $\mathrm{Ap}'(S, n) = (k_1, k_2, \ldots, k_{n-1})$ and $\mathrm{Ap}'(T, n) = (\ell_1, \ell_2, \ldots, \ell_{n-1})$. Apéry tuples give necessary and sufficient conditions for T to be an oversemigroup of S:
$$\begin{aligned} \ell_i + \ell_j &\geq \ell_{i+j} && \text{for } i + j \leq n-1 \\ \ell_i + \ell_j + 1 &\geq \ell_{i+j-n} && \text{for } i + j > n \\ \ell_i &\leq k_i && \text{for } 1 \leq i \leq n-1 \end{aligned}$$

An Apéry tuple uniquely determines a numerical semigroup, so $|O(S)|$ is equal to the number of integer lattice points that lie within the polytope defined by these inequalities.

Ex. $\mathrm{Ap}'(\langle 3, 8\rangle, 3) = (5, 2)$, so the relevant polytope is defined by $x \leq 5,\ y \leq 2,\ 2x \geq y$, and $2y + 1 \geq x$:



There are 10 integer lattice points in this polytope, so $|O(\langle 3, 8\rangle)| = 10$.

Our Results and the Future

By dividing the polytopes with certain hyperplanes we were able to find an explicit formula for $|O(\langle a, b\rangle)|$ for small values of a:

Theorem

For $a = 2, 3,$ and 4 the following formulas hold:
$$\begin{aligned} |O(\langle 2,\ 2k+1\rangle)| &= k + 1 \\ |O(\langle 3,\ 6k+\ell\rangle)| &= (3k+\ell)(k+1) \\ |O(\langle 4,\ 12k+1\rangle)| &= 24k^3 + 30k^2 + 11k + 1 \end{aligned}$$

Let $A(a, b)$ be the total number of (a, b)-cores given by Anderson's Theorem. The fraction of (a, b)-cores from semigroups is thus given by $|O(\langle a, b\rangle)|/A(a, b)$. For small a we can calculate the asymptotic behavior of this:

a	$\lim_{b\to\infty} \lvert O(\langle a, b\rangle)\rvert/A(a, b)$
2	1
3	1/2
4	1/3

This fraction should continue to decrease as a increases. We expect to find the asymptotic behavior of these values for all a.

References

[1] J. Anderson, Partitions which are simultaneously t_1- and t_2-core. Discrete Math. 248 (2002), no. 1–3, 237–243.

[2] P.A. García-Sánchez and J.C. Rosales, Numerical semigroups. New York: Springer, 2009.

[3] N. Kaplan, Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure Appl. Algebra 216 (2012), no. 5, 1016–1032.