Numerical Semigroups and their Corresponding Core Partitions

Benjamin Houston-Edwards
Joint with Hannah Constantin

Yale University

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Background and Review

Definition

A set S is a **numerical semigroup** if

- $S \subseteq \mathbb{N}$
- $0 \in S$
- S is closed under addition
- $\mathbb{N} \setminus S$ is finite

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$S = \langle 3, 8 \rangle = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, \ldots\}$

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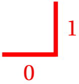

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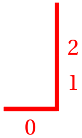

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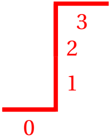

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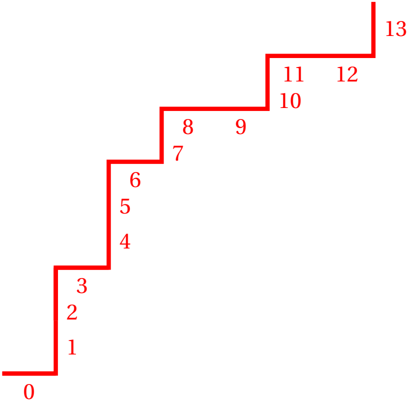

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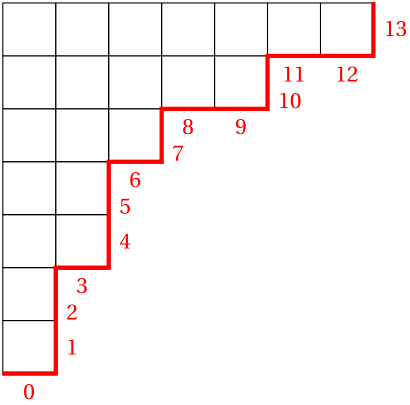

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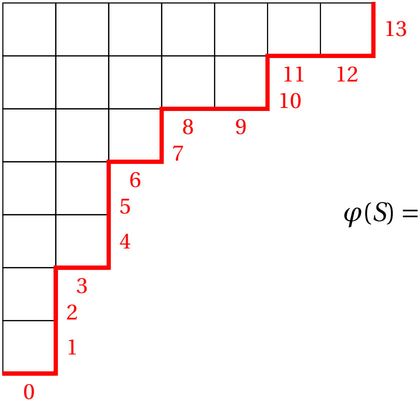

$\varphi(S) = (7, 5, 3, 2, 2, 1, 1)$

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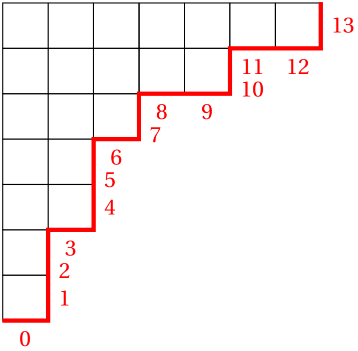

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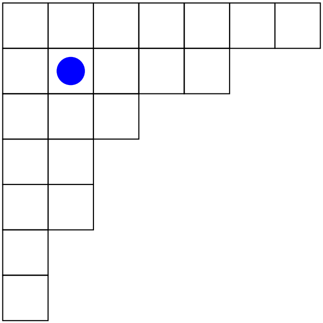

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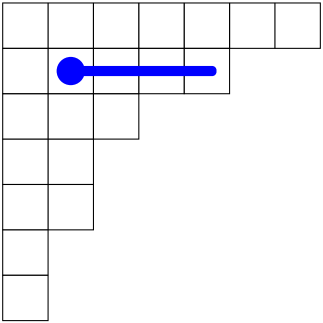

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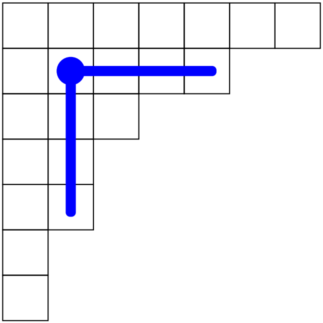

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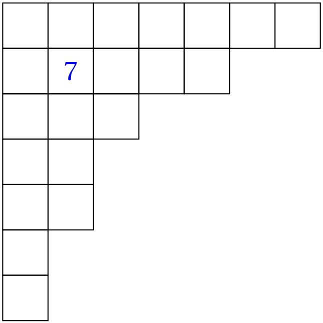

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13	10	7	5	4	2	1
10	7	4	2	1		
7	4	1				
5	2					
4	1					
2						
1						

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Definition

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$\lambda = (7,5,3,2,2,1,1)$ is a $(3,8)-$core

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Theorem (Anderson)

For coprime a and b, the total number of $(a,b)-$core partitions is

$$\frac{1}{a+b}\binom{a+b}{a}.$$

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We are interested in counting the subset of $(a,b)-$cores that come from numerical semigroups via the map φ.

Background

Proposition

Suppose $\lambda = \varphi(S)$ for some semigroup S. Then λ is an $(a, b)-$core if and only if $a, b \in S$.

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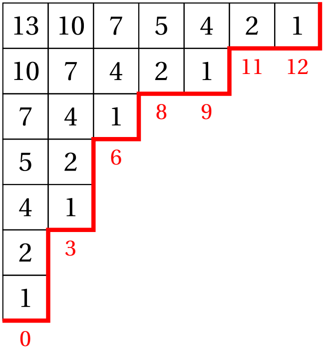
$\lambda = (7,5,3,2,2,1,1)$ is a $(3,8)-$core and $\lambda = \varphi(S)$ where $S = \langle 3,8 \rangle = \{0,3,6,8,9,11,12,14,15,16,\ldots\}$



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Given a numerical semigroup S, the set of **oversemigroups** of S is

$$\{T \supseteq S : T \text{ is a numerical semigroup}\}.$$

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The cardinality of this set is denoted $O(S)$.

The number of $(a, b)-$core partitions from numerical semigroups is exactly $O(\langle a, b\rangle)$.

Apéry Tuples

Definition

If S is a numerical semigroup, then the **Apéry tuple** of S with respect to some $n \in S$ is the tuple $(k_1, k_2, \ldots, k_{n-1})$ such that $nk_i + i$ is the smallest element of S in its residue class $\pmod{n}$ for each i.

This tuple is denoted $\mathrm{Ap}'(S, n)$.

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Example

If $S = \langle 3, 8 \rangle = \{0, 3, 6, 8, 9, 11, 12, 14, \ldots\}$, then 16 and 8 are the smallest elements of S in their residue classes mod 3, so $\mathrm{Ap}'(S, 3) = (5, 2)$.

Apéry Tuples

Suppose S is a numerical semigroup with

$$\mathrm{Ap}'(S,n) = (k_1, \ldots, k_{n-1}).$$

A tuple $(\ell_1, \ell_2, \ldots, \ell_{n-1})$ is an Apéry tuple of some numerical semigroup $T \supseteq S$ if and only if the following inequalities are satisfied:

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$$\begin{aligned}
&\ell_i \geq 0, \qquad \forall 1 \leq i \leq n-1 \\
&\ell_i + \ell_j \geq \ell_{i+j}, \qquad i+j < n \\
&\ell_i + \ell_j + 1 \geq \ell_{n-i-j}, \qquad i+j > n \\
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Remark

These inequalities define an $n-1$ dimensional polytope in which the integer lattice points correspond exactly with the oversemigroups of S.

Apéry Tuples and Polytopes

Example

$S = \langle 3, 8 \rangle$ and $\mathrm{Ap}'(S, 3) = (5, 2)$. The relevant polytope is defined by $x \le 5,\ y \le 2,\ 2x \ge y$, and $2y + 1 \ge x$:

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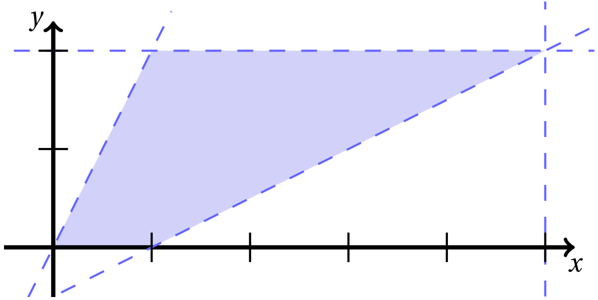

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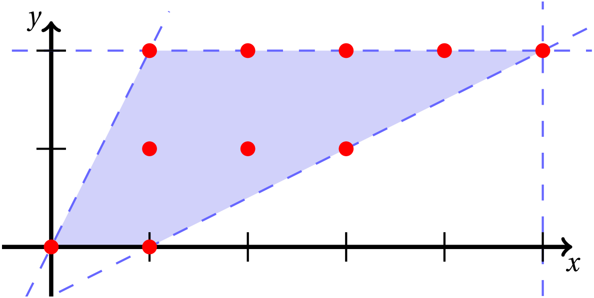
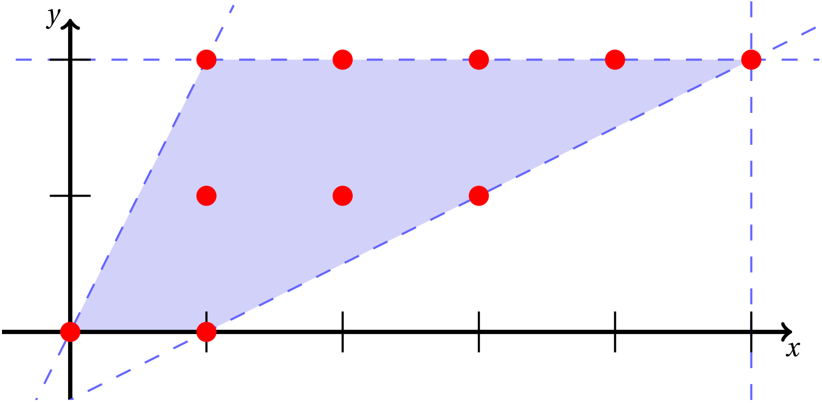


There are 10 integer lattice points in this polytope, so $O(\langle 3, 8 \rangle) = 10$.

The case of $a = 3$



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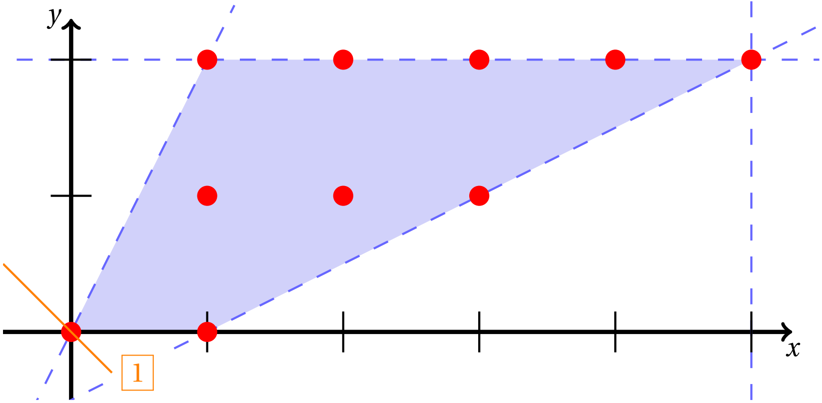

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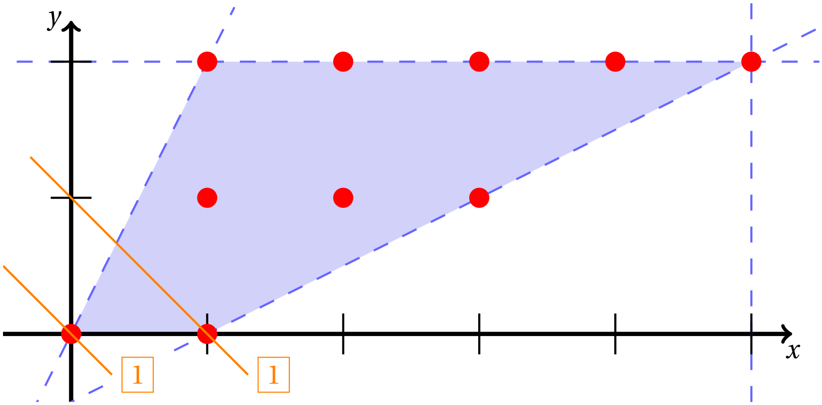

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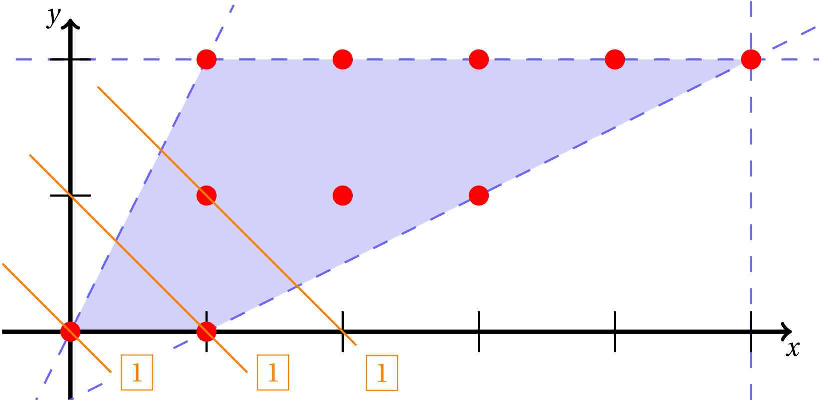

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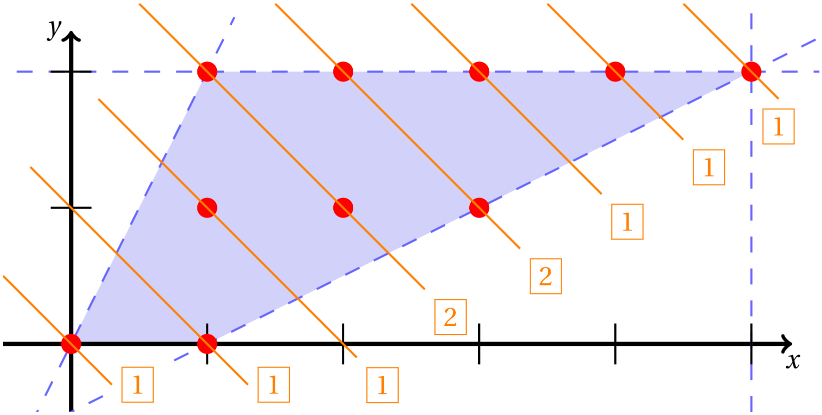

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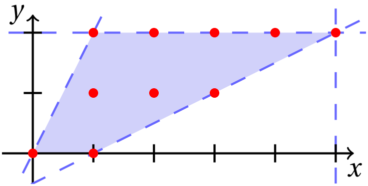

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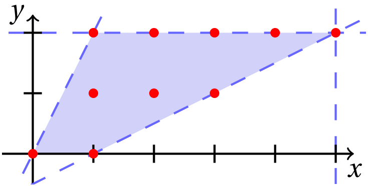

Theorem (Constantin – H.E.)

If $S = \langle 3, 6k + \ell \rangle$ then $O(S) = (3k + \ell)(k + 1)$.

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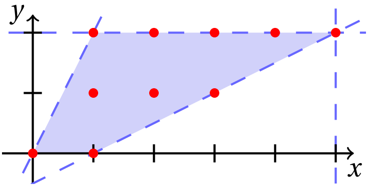

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$O(\langle 3, 8 \rangle) = O(\langle 3, 6 \cdot 1 + 2 \rangle) = (3 + 2)(1 + 1) = 10$

The case of $a = 4$

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In fact, we can find the explicit formula for each ℓ:

ℓ	$O(S)$
1	$24k^3 + 30k^2 + 11k + 1$
3	$24k^3 + 42k^2 + 23k + 4$
5	$24k^3 + 54k^2 + 39k + 9$
7	$24k^3 + 66k^2 + 59k + 17$
9	$24k^3 + 78k^2 + 83k + 29$
11	$24k^3 + 90k^2 + 111k + 45$

Asymptotic Behavior

Let $A(a,b) = \binom{a+b}{a}/(a+b)$, the total number of $(a,b)-$core partitions by Anderson's theorem.

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2	1

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3	1/2

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We suspect that as $a \to \infty$, this fraction will decrease to 0, meaning that almost no $(a,b)-$cores come from semigroups in the limit.

Acknowledgments

We would like the thank . . .

Nathan Kaplan for guiding our research

Flor Orosz Hunziker and Dan Corey for all their help as mentors

Kyle Luh for helping us understand polytopes

The rest of the SUMRY staff and students for creating such a great program

References

J. Anderson, *Partitions which are simultaneously t_1- and t_2-core.* Discrete Math. 248 (2002), no. 1–3, 237–243.

P.A. García-Sánchez and J.C. Rosales, **Numerical semigroups**. New York: Springer, 2009.

N. Kaplan, *Counting numerical semigroups by genus and some cases of a question of Wilf*, J. Pure Appl. Algebra 216 (2012), no. 5, 1016–1032.