Splines mod m

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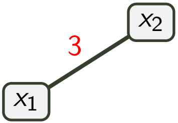
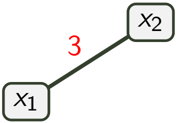
Spline Basics

- Here is a graph with edges labeled with elements of $\mathbb{Z}/27\mathbb{Z}$



- Here is a graph with edges labeled with elements of $\mathbb{Z}/27\mathbb{Z}$
- Can you label the vertices with ring elements x_1 and x_2 so that their difference is a multiple of 3?



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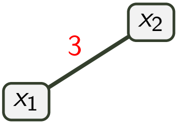
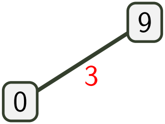


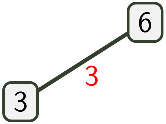
- Of course you can!

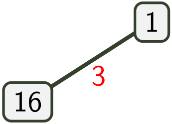
Here's one set of vertex labels you might have found:

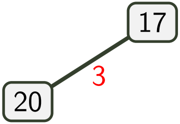


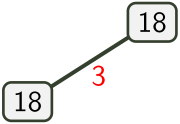
the set of vertex labels $\begin{pmatrix} 9 \\ 0 \end{pmatrix}$ is a spline on the graph.

Here are some other splines on the same graph:

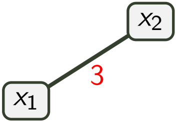








Minimal generating sets are very helpful when talking about splines mod m:



$$\mathbb{B} = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Here is an edge labeled graph.

Here is a minimal generating set for all splines on the edge labeled graph.

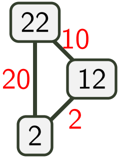
Definition (Spline mod m)

Let G be an edge labeled graph such that the set of edge labels of G is a subset of $\mathbb{Z}/m\mathbb{Z}$. A **spline mod m** is a set of vertex labels in $\mathbb{Z}/m\mathbb{Z}$ that satisfy the following condition:

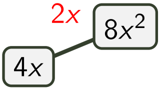
- if two vertices labeled x_1 and x_2 are joined by an edge labeled ℓ_1 then $|x_1 - x_2| \in \langle \ell_1 \rangle$

- We can look for splines on any type of graph.
- We can find splines on graphs labeled with other rings.
- Let's look at a few examples of some other cool splines.

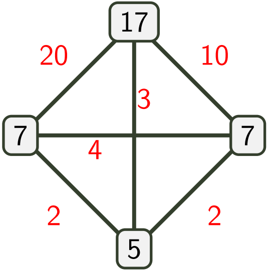
more splines



an integer spline on a 3-cycle



a polynomial spline on one edge

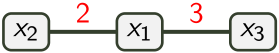


a spline on K_4 in $\mathbb{Z}/30\mathbb{Z}$

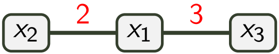
Special Properties (mod m)

Special Properties of Splines mod m

- Finite sets to label with
- Don't label with 0 or units
- Variability of the modulus
- Generating set size

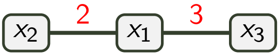


$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}$$



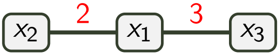
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}$$



$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}/6\mathbb{Z}$$

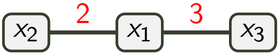
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$



$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}/6\mathbb{Z}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$3 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$



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$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Our minimal generating sets can be very small.

Theorem (Tymoczko, Hagen)

Let G be an edge labeled graph on n vertices. A minimal generating set for **integer** *splines on G must contain exactly n elements.*

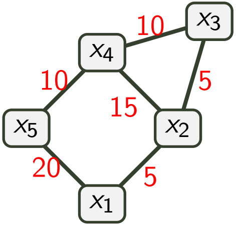
Theorem (Tymoczko, Bowden)

Let G be an edge labeled graph on n vertices. A minimal generating set for **splines mod** *m on G can have anywhere between 1 and n elements.**

- Generating sets are important and they sometimes behave in surprising ways.
- Linear independence can be tricky!
- The value of m matters a lot.

Role of Primes

$\mathbb{Z}/p^2\mathbb{Z}$ example



$$\begin{pmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}/25\mathbb{Z}$$

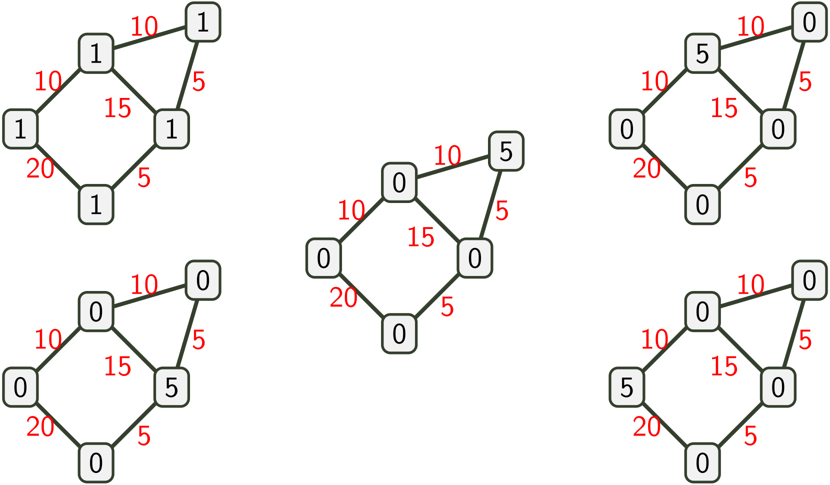
Let's say we want to find a minimal generating set to describe all splines on this graph mod 25...

$\mathbb{Z}/p^2\mathbb{Z}$ theorem

Theorem

Let p be a prime number. If G is a graph on n vertices in $\mathbb{Z}/p^2\mathbb{Z}$, then a minimal generating set for all splines on G is:

$$\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 0 \\ p \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ p \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ . \\ . \\ . \\ p \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} p \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$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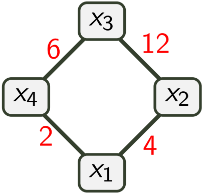
$\mathbb{Z}/32\mathbb{Z}$ example



$$\begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}/32\mathbb{Z}$$

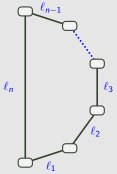
How about all splines on this graph in $\mathbb{Z}/32\mathbb{Z}$?

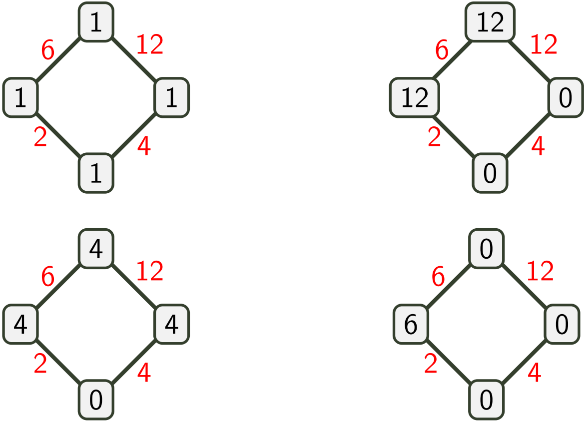
$\mathbb{Z}/p^n\mathbb{Z}$ theorem

Theorem

Let p be a prime number. If C_n is a cycle on n vertices in $\mathbb{Z}/p^k\mathbb{Z}$, then $\mathbb{B}$ is a minimal generating set for all splines on G (up to rotation).

$$\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \ell_1 \\ \ell_1 \\ . \\ . \\ \ell_1 \\ \ell_1 \\ \ell_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_2 \\ \ell_2 \\ . \\ . \\ \ell_2 \\ \ell_2 \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \ell_i \\ . \\ . \\ \ell_i \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \ell_{n-2} \\ \ell_{n-2} \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_{n-1} \\ \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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The Importance of Prime Characterizations

- We are working out a structure theorem that uses the prime factorization of m to understand splines mod m in terms of splines mod p^k.
- This gives an algorithm to compute minimal generating sets.
- In this way $\mathbb{Z}/p^k\mathbb{Z}$ lets us understand more complex modules of splines.

Future Research

Future Research

- Investigate the relationship between graphs and subgraphs.
- Continue to explore variations in minimal generating set size.
- Continue to investigate other moduli.
- Explore, in greater detail, the relationship between splines mod m and splines over other rings.
- Describe all splines over $\mathbb{Z}/p^k\mathbb{Z}$ for arbitrary G

Thank you!

- Thank you to everyone in the math department at Smith for their continued support and guidance.
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