RESULTS ON SOME REAL STABILITY PRESERVERS

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1. Introduction

Real stable univariate polynomials are polynomials which have only real roots. They are the object of much investigation and their theory has had significant progress in recent years. A vast body of research (e.g., [Fis06],[BSS09],[BBb],[BBc]) deals with analyzing the linear transformations that map the set of real stable polynomials to itself. We say that a linear transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ *preserves real stability* if $T(f)$ is real stable, whenever f is real stable. A seminal result by Borcea and Brändén is the complete characterization of all linear transformations that preserve stability of multivariate polynomials [BBb], which in the univariate case amounts precisely to characterization of all linear real stability preservers.

The result by Borcea and Brändén asserts that a particular linear transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserves real stability but does not shed light on where on the real line do the roots of $T(f)$ lie as a function of the roots of f. In this paper we try to answer this question for particular real stability preservers.

First, we examine the movements of the largest and the smallest root after applying linear transformations associated to a multiplier sequence. Then, we shift our attention to another linear stability preserver that plays a role in the recent works of Marcus, Spielman and Srivastava [MSS13a], [MSS13b], viz., the $(1-D)$ operator. We give some preliminary results and derive an asymptotic formula for the gaps between the roots after repeated applications of the $(1-D)$ operator. Specifically, we show that the gaps between roots go to infinity, but under appropriate normalization they converge to the gaps of a particular Hermite polynomial.

2. Multiplier sequences

We focus our attention to a well-studied class of real stability preservers. We examine multiplier sequences and their corresponding linear operators. Conventionally, we would use $\gamma_1, \cdots, \gamma_n$ and $\xi_1, \cdots, \xi_n$ to denote roots of a polynomial, usually ordered in accending order so that γ_1 or ξ_1 is the smallest roots, and γ_n or ξ_n is the largest. We would also write $\gamma_i[f]$ to denote the $i-$th smallest root of f (all polynomials would be real stable, so this is a well-defined function). Finally, the letter λ would be reserved both to denote a multiplier sequnce and to act as a constant in the $(\lambda - D)$ operator. It would be evident from the context which of its meanings is used.

2.1. **Background.** A sequence $\lambda : \mathbb{N} \to \mathbb{R}$ is a *multiplier sequence* if the linear transformation $T_\lambda : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T_\lambda(x^n) = \lambda(n)x^n$ and extended linearly is a stability preserver. For convenience, whenever we write T_λ, we would refer to a linear transformation corresponding to the multiplier sequence λ, even if we have not explicitly said so.

Lemma 1. *[Fis06, Lemma 1.48] Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a multiplier sequence. If $0 \le i \le j \le k$ are such that $\lambda(i)\lambda(k) \neq 0$, then $\lambda(j) \neq 0$. Furthermore, either*

(i) all nonzero $\lambda(i)$ have the same sign, or
(ii) all nonzero entries of the sequence $\{(-1)^i\lambda(i)\}_{i\ge 0}$ have the same sign.

To remain concise, we refer to all multiplier sequences satisfying *(i)* as multiplier sequences of *type 1* and, similarly, these satisfying *(ii)* are called multiplier sequences of *type 2*.

Given two real stable polynomial f and g with roots $\gamma_1 \le \cdots \le \gamma_k$ and $\xi_1 \le \cdots \le \xi_l$, respectively, we say that they *interlace* if $\gamma_1 \le \xi_1 \le \gamma_2 \le \cdots$ or $\xi_1 \le \gamma_1 \le \xi_2 \le \cdots$. It is immediate from the definition that we need either $\deg(f) = \deg(g)$, or $\deg(f) = \deg(g) \pm 1$. Using interlacing ideas one can prove [Fis06, Lemma 6.22]

Proposition 2. *Let λ be a multiplier sequence of type 1. If f is a real stable polynomial with roots $\gamma_1 \le \cdots \le \gamma_n$, then*

$$k\text{-th biggest root of } \left(T_\lambda\left[(x-\lambda_k)^k\right]\right) \ge k\text{-th biggest root of } (T_\lambda(f)).$$

Proposition 3. *Let λ be a multiplier sequence of type 2. If f is a real stable polynomial with roots $\gamma_1 \leq \cdots \leq \gamma_n$, then*

$$k\text{-th biggest root of } (T_\lambda(f)) \geq k\text{-th biggest root of } \big(T_\lambda\big[(x-\lambda_k)^k\big]\big).$$

We are interested mainly in the movement of the smallest and the largestl roots, in which case other techniques give sharper estimates.

2.2. **Majorization.** Given two vectors with weakly increasing entries $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we say that *x majorizes y*, $x \succeq y$, if $\sum_1^n x_i = \sum_1^n y_i$ and $\sum_k^n x_i \geq \sum_k^n y_i$ for $2 \leq k \leq n-1$. Since we can associate each polynomial to the weakly increasing vector of its roots, we say that *f majorizes g*, $f \succeq g$, if f and g are real stable, share the same leading coefficient, and satisfy $x \succeq y$ where x and y are the weakly increasing vectors consisting of the roots of f and g, respectively. Notice that if $f \succeq g$, they have the same leading and second highest coefficients, max root$(f) \geq$ max root (g), and min root$(g) \geq$ min root (f). We also say that a linear transformation T *preserves majorization* if $T(f) \succeq T(g)$ whenever $f \succeq g$. An important result which is a special case of a theorem by Borcea and Brändén is that T_λ preserves majorization for every multiplier sequence λ [BBa, Theorem 1]. This fact would allow us to give more accurate bounds on the movement of the largest and smallest roots of a real stable polynomial.

2.3. **First bounds.** Note that if $f = \sum_{k=0}^n a_k x^k$ is real stable, then $f \succeq a_n(x + \frac{a_{n-1}}{na_n})^n$. Applying the T_λ operator, we derive the following

Proposition 4. *Suppose $f = \sum_{i=0}^n a_i x^i$ is a real stable polynomial. Then for any multiplier sequence λ the following inequalities holds*

$$\gamma_{max}[T_\lambda(f)] \geq -\frac{a_{n-1}}{na_n}\gamma_{max}\big[T_\lambda((1-x)^n)\big],$$

$$-\frac{a_{n-1}}{na_n}\gamma_{min}\big[T_\lambda((1-x)^n)\big] \geq \gamma_{min}[T_\lambda(f)].$$

Proof. The above remark, coupled with the result by Borcea and Brändén, yields to $T_\lambda(f) \succeq T_\lambda(a_n(x+\frac{a_{n-1}}{na_n})^n)$. The linearity of T_λ leads to the equality $T_\lambda(a_n(x + \frac{a_{n-1}}{na_n})^n) = \frac{a_{n-1}}{na_n} T_\lambda(a_n(\frac{x}{\frac{a_{n-1}}{na_n}} + 1)^n)$ and so

$$\gamma_{max}\big[T_\lambda(a_n(x+\frac{a_{n-1}}{na_n})^n\big] = \frac{a_{n-1}}{na_n}\gamma_{max}\big[T_\lambda((x+1)^n)\big] = -\frac{a_{n-1}}{na_n}\gamma_{max}\big[T_\lambda((x-1)^n)\big].$$

Since $f \succeq g$ implies $\gamma_{max}[f] \geq \gamma_{max}[g]$ and $\gamma_{min}[g] \geq \gamma_{min}[f]$, the statement follows. $\square$

In the case of the multiplier sequence $\lambda(n) = 1/n!$, this result reduces to

Proposition 5. *For a real stable polynomial $f = \sum_{i=0}^n a_i x^i$, we have*

$$\gamma_{max}[T_!(f)] \geq -\frac{a_{n-1}}{na_n}\gamma_{max}[L_n(x)],$$

$$-\frac{a_{n-1}}{na_n}\gamma_{min}[L_n(x)] \geq \gamma_{min}[T_!(f)],$$

where $L_n(x)$ is the n-th Laguerre polynomial.

Taking into consideration the signs of the roots, Proposition 4 can be tightened to

Proposition 6. *Given a real stable polynomial f with m nonnegative and n negative roots, denote by σ_+ the sum of the nonnegative roots of f and by σ_- the sum of negative ones. Then, for any multiplier sequence λ, we have*

$$\gamma_{max}[T_\lambda(f)] \geq \gamma_{max}\big[(x - \frac{\sigma_+}{m})^m (x - \frac{\sigma_-}{n})^n\big],$$
$$\gamma_{min}\big[(x - \frac{\sigma_+}{m})^m (x - \frac{\sigma_-}{n})^n\big] \geq \gamma_{min}[T_\lambda(f)],$$

Proof. It is immediate from the definition of majorization that for two pair of vectors $x \succeq y$ and $u \succeq v$, satisfying $\max(x_i) \leq \min(u_i)$ and $\max(y_i) \leq \min(v_i)$, it holds that $(x, u) \succeq (y, v)$. Let v_1 be the weakly increasing vector of the negative roots of f, and v_2 – of the positive. The result now follows from $v_1 \succeq (\frac{\sigma_-}{n}, \cdots, \frac{\sigma_-}{n})$ and $v_2 \succeq (\frac{\sigma_+}{m}, \cdots, \frac{\sigma_+}{m})$. □

2.4. **Second bounds.** Given a real stable f, we find a polynomial that majorizes it, which would give give another bound on the movements of the smallest and largest roots. Nevertheless, we are interested in preserving as much information possible so that the bounds obtained do not differ significantly from the actual movement of the largest and smallest roots. For that reason we classify real stable polynomials based on the signs of their roots.

Definition 7. *A real stable polynomial $f \in \mathbb{R}[x]$ is* of type (p_+, p_-, p_0) *if f has p_+ positive, p_- negative and p_0 roots, respectively.*

Extracting the type of real stable polynomial is not a hard task. Indeed, one needs only to know the signs of the coefficients of the polynomial f, which can be seen as a consequence of Descartes' rule of signs.

Definition 8. *Linear transformation $T : \mathbb{R}[x] \mapsto \mathbb{R}[x]$* preserves the type *if $T(f)$ is of the same type as f for any real stable f.*

A natural question is how does a multiplier sequence affect the triple (p_+, p_-, p_0). Surprisingly, in a lot of cases it does not.

Proposition 9. *Suppose λ is a multiplier sequence of type 1. Then T_λ preserves the type.*

Proof. Follows directly from Descartes' rule of signs. □

For a real stable polynomial f we find another polynomial based on the type of the former that majorizes it.

Lemma 10. *Let $f = \sum_{k=1}^n a_k x^k$ be polynomial of type (p_+, p_-, p_0) with roots $\{\gamma_i\}_{i=1}^n$. Then*

$$(x - M)^{p-1}(x - M + \frac{a_{n-1}}{a_n})(x + M)^p x^{n-2p} \succeq P(x), \tag{1}$$

where $p = \min(p_+, p_-)$ and $M = \max |\gamma_k(P)|$.

Proof. We only need to show that

$$(-M, -M, \cdots, -M, 0, \cdots, 0, M, \cdots, M - \frac{a_{n-1}}{a_n}) \succeq (\gamma_1, \cdots, \gamma_n).$$

There are two cases, $p_+ \geq p_-$ and $p_+ < p_-$. In both cases, we check the majorization directly from the definition. □

For convenience, we denote the polynomial on the left hand side of Equation 1 by H_f. From the above lemma we naturally derive

Proposition 11. *Suppose f is a real stable polynomial. Then, for any multiplier sequence λ, the following inequalities hold*

$$\gamma_{max}\big[T_\lambda(H_f)\big] \geq \gamma_{max}[T_\lambda(f)]$$
$$\gamma_{min}[T_\lambda(f)] \geq \gamma_{min}\big[T_\lambda(H_f)\big].$$

2.5. **Bounding $T_!(H_f)$.** Analyzing $T_\lambda(H_f)$ for arbitrary λ seems a complicated task. However, in the case $\lambda(n) = \frac{1}{n!}$, we may use a notion recently introduced by Batson, Spielman, and Srivastava in their [BSS09] to provide a sharp upper bound on $T_!(H_P)$. They introduce the *lower* and *upper barrier functions* of a real stable polynomial f as $\Phi_f(b) = -f'(b)/f(b)$ and $\Phi^f(b) = f'(b)/f(b)$, respectively, as well as the associated quantities

$$\mathrm{smin}_\varphi(f) := \min\{x \in \mathbb{R} : \Phi_f(b) = \varphi\},$$
$$\mathrm{smax}_\varphi(f) := \max\{x \in \mathbb{R} : \Phi^f(b) = \varphi\}.$$

Modifications of proofs by Marcus, Spielman and Srivastava for the $(1 - D)$ operator, where D stands for the differential operator d/dx, yields the following

Lemma 12. *Let f be a real stable polynomial, $\varphi > 0$, and $0 \neq \lambda \in \mathbb{R}$. Then*

$$\mathrm{smin}_\varphi\left((\lambda - D)f\right) \geq \mathrm{smin}_\varphi(f) + \frac{1}{\lambda + \varphi}$$

holds whenever $\lambda > 0$ or $\varphi < -\lambda$.

Proof. In Chris's notes on the barrier function. □

Lemma 13. *Suppose $\lambda \in \mathbb{R}$. If f has real roots and $0 < \varphi < |\lambda|$, then*

$$\mathrm{smax}_\varphi\left((\lambda - D)f\right) \leq \mathrm{smax}_\varphi(f) + \frac{1}{\lambda - \varphi}.$$

Proof. Similar to above. □

Perhaps at first the connection between the $T_!$ and the $(\lambda - D)$ operator is not evident. Nevertheless, the proof of the real stability of the $(1 - D)$ operator, which is outlined in the next section, establishes the following relation between the two

Lemma 14. *For a real stable polynomial f with roots $\{\gamma_i\}_{i=1}^n$, we have*

$$T_!(f) = [\prod_{\gamma_i \neq 0}(1/\gamma_i - D)\prod_{\gamma_i = 0} D]x^n,$$

where the multiplication $T_1 T_2$ of two operators is their composition.

We are now equipped with all tools to state the following

Theorem 15. *Let $f = \sum_{k=1}^{n} a_k x^k$ be polynomial of type (p_+, p_-, p_0) with roots $\{\gamma_i\}_{i=1}^n$ that have positive sum. Then*

$$M\Big(\min_{0<\varphi<c} \frac{p-1}{1-\varphi} - \frac{p}{1+\varphi} + \frac{1}{c-\varphi} + \frac{n}{\varphi}\Big) \geq \gamma_{max}(T_!(f)),$$

$$M\Big(\max_{0<\varphi<1} \frac{p-1}{1+\varphi} + \frac{p}{\varphi-1} + \frac{1}{c+\varphi} - \frac{n}{\varphi}\Big) \leq \gamma_{min}(T_!(f)),$$

where $p = \min(p_+, p_-)$, $c = \frac{1}{1-\frac{a_{n-1}}{a_n M}}$ with $M = \max|\gamma_k|$.

Proof. Due to linearity of $T_!$, we derive $\gamma_{max}[T_!(H_f)] = M\gamma_{max}[T_!((x^2-1)^p(x-\frac{1}{c})x^{n-2p-1}]$, where c is as above. Lemma 14 yields

$$T_!((x-1)^{p-1}(x+1)^p(x-c)x^{n-2p}) = [(1-D)^{p-1}(1+D)^p(c-D)D^{n-2p}]x^n.$$

Using the bounds for smax_φ and the fact that D moves the largest root to the left, we obtain that

$$\gamma_{max}\big[T_!((x-1)^{p-1}(x+1)^p(x-c)x^{n-2p})\big] \leq \frac{p-1}{1-\varphi} - \frac{p}{1+\varphi} + \frac{1}{c-\varphi} + \frac{n}{\varphi}$$

holds for every positive $\varphi < c$. Similarly, since D moves the smallest root to the right, we derive $\frac{p-1}{1+\varphi} + \frac{p}{\varphi-1} + \frac{1}{c+\varphi} - \frac{n}{\varphi} \leq \gamma_{min}[T_!(f)]$ for every positive $\varphi < 1$. □

Since for the movement of the minimal root after application of $T_!$, we do not need to impose any conditions on c, we prove

Proposition 16. *Let $f = \sum_{k=1}^{n} a_k x^k$ be a polynomial of type (p_+, p_-, p_0) with roots $\{\gamma_i\}_{i=1}^n$ that have positive sum. Then*

$$-M(\sqrt{n} + \sqrt{p-1})^2 + (p - \frac{1}{2})M \leq \gamma_{min}[T_!(f)],$$

where $M = \max|\gamma_i|$ and $p = \min(p_+, p_-)$.

Proof. Choosing $\varphi = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{p-1}}$ in Theorem 15 gives

$$M(\sqrt{n}+\sqrt{p-1})(\frac{p}{2\sqrt{n}+\sqrt{p}} + \frac{1}{(c+1)\sqrt{n}+c\sqrt{p}} - \sqrt{n} - \frac{p}{\sqrt{p-1}}) \leq \gamma_{min}(T_!(f))$$

Note that $(\sqrt{n} + \sqrt{p-1}) \geq \frac{1}{2}((c+1)\sqrt{n} + c\sqrt{p})$ since $c \leq 1$, and that $\frac{p}{\sqrt{p-1}} \geq \sqrt{p-1}$. Clearly, we also have $2(\sqrt{n} + \sqrt{p-1}) \geq 2\sqrt{n} + \sqrt{p-1}$. Therefore, the left hand side of the inequality is greater than

$$-M(\sqrt{n} + \sqrt{p-1})^2 + (p - \frac{1}{2})M,$$

which is what we wanted to prove. □

3. $(\lambda - D)$ Operators

3.1. **Background.** In addition to being critical in Marcus, Spielman, and Srivastava's recent works ([MSS13a],[MSS13b],[**?**]) on the Bourgain-Tzafriri restricted invertibility theorem and the Weaver conjecture, operators of the form $(\lambda - D)$ for $\lambda \in \mathbb{R}$ can be used to understand a large class of linear stability preservers. For example, suppose we have a polynomial f with real roots $\gamma_1, \ldots, \gamma_n$. Then f can be factored as $f(x) = c(\gamma_1 - x) \cdots (\gamma_n - x)$ for some $c \in \mathbb{R}$. We can see, then, that

$$f(D) = c(\gamma_1 - D) \cdots (\gamma_n - D)$$

is a linear stability preserver made up of $(\lambda - D)$ operators. Understanding operators of this form can also help study an even larger class of operators, including multiplier sequences.

3.2. **Basic Results.** First we provide a proof that $(\lambda - D)$ actually preserves real stability. This is easiest to do using the following lemma:

Lemma 17. *Suppose f and g are two real stable polynomials with interlaced roots. Then for any $r, s \in \mathbb{R}$, $rf + sg$ is real stable.*

Proof. Suppose without loss of generality that the degree of f is at least the degree of g. It follows from the interlacing that $rf + sg$ alternates sign at successive roots of f. By intermediate value theorem, then, there is a root between each of the roots of f. This gives that there are at least $\deg f - 1$ real roots. Because complex roots of polynomials in $\mathbb{R}[x]$ come in pairs, the remaining root must be real as well. Details can be found in [Fis06, p. 20]. □

Because Df interlaces f (by Rolle's Theorem), it follows immediately that $(\lambda - D)$ preserves real stability.

Now we begin to address the question of what exactly happens to the roots under transformation. The following result is perhaps the easiest to see:

Proposition 18. *The $(\lambda - D)$ operator increases the average root of a polynomial by $1/\lambda$.*

Proof. From Vieta's formulas, the second highest coefficient divided by the highest is minus the sum of the roots, so that if $f(x) = \sum_{i=0}^{d} a_i x^i$ the average root of f is $-\frac{a_{d-1}}{da_d}$. Applying $(\lambda - D)$, the new average root is

$$-\frac{\lambda a_{d-1} - da_d}{d\lambda a_d} = -\frac{a_{d-1}}{da_d} + \frac{1}{\lambda}.$$ □

A tool which will prove useful in much of the remaining analysis is the *logarithmic derivative*, defined as $\frac{Df}{f}$ for a function f. The name results from the fact that $\frac{d}{dx} \log f = \frac{Df}{f}$. This is useful because it is zero at the roots of

Df (except those which are also roots of f), and has a nice expression in terms of the roots of f: if $\gamma_1, \ldots, \gamma_m$ are the roots of f then

$$\frac{Df}{f}(x) = \sum_{i=1}^{m} \frac{1}{x - \gamma_i}. \tag{2}$$

We can use this to quickly show which direction the roots move.

Proposition 19. *The $(\lambda - D)$ operator moves each root to the right if $\lambda > 0$, and to the left if $\lambda < 0$ (roots may also be unchanged in either case).*

Proof. Let x be the smallest root of $(\lambda - D)f$. Then it is either a root of f or a root of $\frac{(\lambda - D)f}{f}$. If it is a root of the latter we have

$$\lambda = \frac{Df}{f}(x) = \sum_{i=1}^{m} \frac{1}{x - \gamma_i}.$$

If x is below the roots of f, then the sum is negative so the equation cannot hold if $\lambda > 0$. Therefore the minimum root of $(\lambda - D)f$ is at least the minimum root of f. Since f and $(\lambda - D)f$ interlace, the other roots must increase as well. A similar argument gives the result for $\lambda < 0$. $\square$

We can also use the logarithmic derivative to prove the following theorem, found in [FR04] and attributed to M. Riesz [Sto25]:

Theorem 20. *Let f be a real stable polynomial with roots $\gamma_1 \leq \cdots \leq \gamma_m$, and let $\delta = \min_i(\gamma_{i+1} - \gamma_i)$. Similarly, let $\xi_1 \leq \cdots \leq \xi_{m-1}$ be the roots of $(\lambda - D)f$ and let $\Delta = \min_i(\xi_{i+1} - \xi_i)$. Then $\Delta \geq \delta$.*

Proof. First, notice that if f has a multiple root then $\delta = 0$, and since $\Delta \geq 0$ the theorem is trivial. Assume instead that all roots of f are simple (so that f and $(\lambda - D)f$ share no roots). We will prove the theorem by contradiction: suppose there were two adjacent roots $p > q$ of $(\lambda - D)f$ that were closer together than any two roots of f, i.e.

$$p - q < \gamma_{i+1} - \gamma_i \quad \forall i.$$

We can rearrange this inequality to give

$$\frac{1}{p - \gamma_{i+1}} > \frac{1}{q - \gamma_i}$$

since the denominators always have the same sign. Because p, q are also roots of $\frac{(\lambda - D)f}{f} = \lambda - \frac{Df}{f}$, it follows that $\frac{Df}{f}(p) - \frac{Df}{f}(q) = 0$. From Equation 2, we can write this as

$$0 = \sum_{i=0}^{m} \left(\frac{1}{p - \gamma_i} - \frac{1}{q - \gamma_i} \right) = \sum_{i=0}^{m-1} \left(\frac{1}{p - \gamma_{i+1}} - \frac{1}{q - \gamma_i} \right) + \frac{1}{p - \gamma_1} - \frac{1}{q - \gamma_m}.$$

However, every term is strictly greater than zero. Therefore we have derived a contradiction and the theorem must hold. $\square$

By bounding the sum in Equation 2 we can derive bounds on the movement of particular roots:

Proposition 21. *Let $\gamma_i[f]$ denote the i-th smallest root of f. Then if $\lambda > 0$*
$$\gamma_i[(\lambda - D)f] - \gamma_i[f] \leq \frac{i}{\lambda},$$
and if $\lambda < 0$
$$\big|\gamma_i[(\lambda - D)f] - \gamma_i[f]\big| \leq \frac{m - i + 1}{|\lambda|}.$$

Proof. If $x := \gamma_i[(\lambda - D)f]$ is a root of f then both differences above are zero and we are done. Assume instead that x is a root of
$$\frac{(\lambda - D)f}{f}(x) = \lambda + \sum_{j=1}^{m} \frac{1}{\gamma_j[f] - x}.$$
If $\lambda > 0$ then $\gamma_i[f] < x < \gamma_{i+1}[f]$, so we can bound the sum to give
$$0 = \frac{(\lambda - D)f}{f}(x) > \lambda + \frac{i}{\gamma_i[f] - x}$$
so
$$x - \gamma_i[f] < \frac{i}{\lambda}.$$
If $\lambda < 0$ then $\gamma_{i-1}[f] < x < \gamma_i[f]$, so we can bound the sum to give
$$0 < \lambda + \frac{m - i + 1}{\gamma_i[f] - x}$$
so $x - \gamma_i[f] > \frac{m-i+1}{\lambda}$. □

Both results in the above proposition give *upper* bounds on the absolute movement of a root. For the largest and smallest roots we can also give lower bounds:

Proposition 22. *If $\lambda > 0$,*
$$\gamma_m[(\lambda - D)f] - \gamma_m[f] \geq \frac{1}{\lambda}.$$
If $\lambda < 0$,
$$\big|\gamma_1[(\lambda - D)f] - \gamma_1[f]\big| \geq \frac{1}{|\lambda|}.$$

The proofs are essentially the same as in the previous proposition. Combining these results yields the following:

Theorem 23. *For any $\lambda \neq 0$, $(\lambda - D)$ increases the average root spacing.*

Proof. If $\lambda > 0$, from Proposition 21 the smallest root is increased by at most $\frac{1}{\lambda}$. From Proposition 22, the maximum root is increased by at least $\frac{1}{\lambda}$. Therefore the difference between the maximum and minimum roots, and hence the average spacing, increases. Similarly for $\lambda < 0$. □

It is worth mentioning a heuristic model for the effect of $(\lambda - D)$ on a polynomial's roots: recall that roots of $(\lambda - D)f$ which are not roots of f are solutions to

$$-\lambda + \sum_{i=0}^{m} \frac{1}{x - \gamma_i} = 0.$$

We can interpret each term on the left as a force on a particle at x, so that solutions are the positions of equilibrium. The forces are a constant force of 'gravity' given by $-\lambda$ (negative force pushes in the $-x$ direction) and a collection of unit 'electric charges' fixed at each of the roots of f[1]. The roots of $(\lambda - D)$ are then the places on the real line where the repulsive forces due to the charges and the force of gravity cancel each other. A similar model for the differential operator was known to Gauss; this particular model is from [BSS09].

3.3. **Asymptotic Root Distribution.** For simplicity, we will now temporarily focus on the special case of the $(1 - D)$ operator. The following lemma proves something one might expect from Gauss's model: that the roots tend to 'repel' each other through the action of $(1 - D)$.

Lemma 24. *Suppose f is a polynomial whose ith smallest root γ_i is moved (necessarily to the right) a distance δ by the $(1-D)$ operator. Given $\mu \in \mathbb{R}$, consider the polynomial $\tilde{f} = (x - \mu)f$. Then*

(1) if $\mu \le \gamma_i$, then $\gamma_{i+1}\big[(1 - D)\tilde{f}\big] \ge \gamma_i + \delta$ (note that since a root has been added to the left, the $(i + 1)$th root of $\tilde{f}$ is the root which one might intuitively identify with the ith root of f), and

(2) if $\mu > \gamma_i$, then $\gamma_i\big[(1 - D)\tilde{f}\big] \le \gamma_i + \delta$.

Loosely speaking, the added root has the effect of 'repelling' the old roots of $(1 - D)f$.

Proof. First consider the second case, $\mu > \gamma_i$. If $\mu \le \gamma_i + \delta$, then the result follows immediately from interlacing.

If instead $\mu > \gamma_i + \delta$, we can use the logarithmic derivative to get the bound we want. If $\delta = 0$ then γ_i is a root of multiplicity and adding the root μ will not change how much it moves. Assume instead then that $\delta > 0$. Then $\gamma_i + \delta$ is a root of $\frac{f'}{f}$, i.e.

$$\sum_{j=1}^{n} \frac{1}{(\gamma_i + \delta) - \gamma_j} = 1.$$

We would like to find the root of $(1 - D)\tilde{f}$, that is the solution of

$$\sum_{j=1}^{n} \frac{1}{x - \gamma_j} + \frac{1}{x - \mu} = 1,$$

[1]Strictly speaking, the force due to an actual electric charge would be proportional to $(\text{distance})^{-2}$ instead of $(\text{distance})^{-1}$, but it's close enough for a heuristic model.

that is closest to the right of γ_i. We know that for $x = \gamma_i + \delta$ the sum is equal to 1 and the second term is negative, so the left hand side is less than 1. It is also apparent that the sum is much greater than 1 for x just to the right of γ_i, and continuous for $x \in (\gamma_i, \gamma_i + \delta)$. Therefore by the intermediate value theorem there is a solution in $(\gamma_i, \gamma_i + \delta)$ and the result immediately follows.
The proof for $\mu < \gamma_i$ is essentially the same. $\square$

The following lemma easily follows:

Lemma 25. *Appending a root between two (not necessarily adjacent) roots increases the change in their difference upon applying the* $(1 - D)$ *operator.*

For example, if the gap between the roots of $(x - 1)(x - 3)$ changes by δ, then the gap between the outer two roots of $(x - 1)(x - 2)(x - 3)$ will change by more than δ.

We already know that the gap between the largest and smallest roots is nondecreasing. The previous lemma can be used to show that it approaches infinity:

Lemma 26. *For any polynomial* f*,*

$$\lim_{n\to\infty} \gamma_{\max}\big[(1 - D)^n f\big] - \gamma_{\min}\big[(1 - D)^n f\big] = \infty.$$

Proof. Because the quantity of interest is nondecreasing (from Theorem 23), it is enough to show that it has no finite limit. Suppose it did approach some number $l < \infty$. Then there is some number of iterations of $(1 - D)$ after which the difference between the maximum and minimum roots is arbitrarily close to l. Suppose a polynomial with only two roots separated by l has its gap increased by δ. Then if the two roots are closer than l and more roots are appended in between, the gap will still increase by at least δ. Therefore if the gap between extremal roots is at least $l - \delta/2$, then after applying $(1 - D)$ it will be at least $l + \delta/2$. This is a contradiction, so it must be that the gap is unbouded. $\square$

Using this and the following lemma, we will be able to prove that all root gaps go to infinity. First a definition:

Definition 27. *An* m-gap *or* gap of degree m *is a gap between two roots that have* $m - 1$ *other roots between them. For example* $\gamma_4 - \gamma_1$ *is a 3-gap, and the difference between the minimum and maximum roots of a degree* d *polynomial is an* $(d - 1)$*-gap.*

Now the lemma:

Lemma 28. *Suppose the* m*-gaps of a given polynomial approach infinity. Then the* $m - 1$ *gaps do as well.*

Proof. Suppose the $(m - 1)$-gaps of a polynomial f of degree d do *not* approach infinity. Then the limit inferior of the smallest $(m - 1)$-gap is finite, say l. We can write this as

$$\liminf_{n\to\infty} G^{(m-1)}_{\min}\big[(1-D)^n f\big] = l$$

where $G^{(m)}_{\min}[f]$ is the size of the smallest m-gap of f. Choose ε so that two (isolated) roots with separation less than $l+\varepsilon$ will have their separation increased by at least 3ε. Choose L large enough that

(1) If an $(m-1)$-gap is at most $l+d+\varepsilon$ wide, and all roots outside the gap are at least L away, then the size of the gap increases; and
(2) If an $(m-1)$-gap is at most $l+\varepsilon$ wide, and all roots outside the gap are at least L away, then the outside roots' effect on the change in gap size is at most ε.

By the assumption that the m-gaps approach infinity, there exists N so that for all $n > N$

$$G^{(m)}_{\min}\big[(1-D)^n f\big] > L + l + d + \varepsilon.$$

By the assumption that the limit inferior of the smallest $(m-1)$-gap is l, there exist arbitrarily large n such that

$$l - \varepsilon < G^{(m-1)}_{\min}\big[(1-D)^n f\big] < l + \varepsilon.$$

Choose such an n which is greater than N. Due to the choice of ε and N, we then have

$$G^{(m-1)}_{\min}\big[(1-D)^{n+1} f\big] > l + \varepsilon.$$

For all further iterations, we have that an $(m-1)$-gap either increases, or if it is larger than $l+d+\varepsilon$, it may decrease. However, it may only decrease by at most d (since by Proposition 21 the i-th smallest root moves by at most i). Therefore the size of an $(m-1)$-gap will never drop below $l+\varepsilon$, contradicting that l is the limit inferior. It must then be that the limit inferior is infinite, so the size of the smallest $(m-1)$-gap approaches infinity. □

Lemmas 26 and 28 together immediately give

Theorem 29. *If f is any polynomial, the difference between any pair of roots of $(1-D)^n f$ approaches infinity as n approaches infinity.*

Proof. Induction on gap degree using Lemma 28, with Lemma 26 as the base case. □

However, in a sense the roots tend to converge on their average, as given by the following theorem.

Theorem 30. *If f is any polynomial, then for any i the ratio of the ith root of $(1-D)^n f$ to the average root approaches 1 as n approaches infinity.*

Proof. Because the coefficients of a polynomial are the symmetric polynomials in its roots, given $f(x) = \sum a_i x^i$ with roots $\gamma_1, \cdots, \gamma_m$ and average root γ_{avg} it follows that the polynomial

$$\sum_{i=0}^{m} \frac{a_i}{\gamma_{\text{avg}}^{m-i}} x^i$$

has roots $\gamma_i/\gamma_{\text{avg}}$. Therefore we want to show that as n approaches infinity, the polynomial obtained by dividing the ith coefficient of $(1-D)^n f$ by the $(m-i)$th power of its average root approaches a constant multiple of $(x-1)^m$.

$(1-D)$ always increases the average root by 1 (from Proposition 18), so

$$\gamma_{\text{avg}}[Đ^n f] = \gamma_{\text{avg}} + n.$$

We now just need to calculate the coefficients after n iterations. Because $(1-D)$ is linear we first restrict our attention to the coefficients of $(1-D)^n x^i$. Using the binomial theorem, we can calculate

$$(1-D)^n x^i = \sum_{j=0}^{n} \binom{n}{j} 1^j (-D)^{n-j} x^i = \sum_{j=n-i}^{n} \binom{n}{j} (-1)^{n-j} \frac{i!}{(i-(n-j))!} x^{i-(n-j)}.$$

Using Đ for $(1-D)$, we can notate this as

$$Đ^n[x^i](x) = \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \frac{n!}{(n-(i-j))!} x^j.$$

The coefficient of the x^k term is then

$$\binom{i}{k} \sum_{j=0}^{i-k} (-1)^j \left[\begin{array}{c} i-k \\ j \end{array} \right] n^j,$$

where $\left[\begin{array}{c} n \\ k \end{array} \right]$ is an unsigned Stirling number of the first kind, so

$$Đ^n[x^i](x) = \sum_{k=0}^{i} \left[\binom{i}{k} \sum_{j=0}^{i-k} (-1)^j \left[\begin{array}{c} i-k \\ j \end{array} \right] n^j \right] x^k.$$

We can now use this to calculate

$$Đ^n[f](x) = \sum_{i=0}^{m} a_i \left[\sum_{k=0}^{i} \left[\binom{i}{k} \sum_{j=0}^{i-k} (-1)^j \left[\begin{array}{c} i-k \\ j \end{array} \right] n^j \right] x^k \right].$$

To divide each root by $\gamma_{\text{avg}}[Đ^n f] = \gamma_{\text{avg}} + n$, we divide the polynomial by $(\gamma_{\text{avg}} + n)^m$ and replace x with $(\gamma_{\text{avg}} + n)x$ to get

$$\sum_{i=0}^{m} a_i \left[\sum_{k=0}^{i} \left[\binom{i}{k} \sum_{j=0}^{i-k} (-1)^j \left[\begin{array}{c} i-k \\ j \end{array} \right] \frac{n^j}{(\gamma_{\text{avg}} + n)^{m-k}} \right] x^k \right].$$

If we now let n approach infinity, all terms will go to zero except where $j = m-k$. This happens exactly when $i = m$ and $j = i-k$, so the limit is

$$a_m \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left[\begin{array}{c} j \\ j \end{array} \right] x^k = a_m (x-1)^m,$$

which is what we wanted. ∎

Above we gave a more intuitive proof that all root gaps go to infinity motivated by Gauss's electric charge model. It is also possible to show this purely algebraically using a method similar to that of Theorem 30, while giving asymptotics for each root gap. To be clear, we first give a definition:

Definition 31. *For a polynomial f with real roots $\gamma_1 \leq \cdots \leq \gamma_d$, the* ith root gap *of f is $G_i[f] := \gamma_{i+1} - \gamma_i$.*

Now we can state and prove the following:

Theorem 32. *For any degree d polynomial f,*

$$G_i[(1-D)^n f] \sim \sqrt{n} \cdot G_i[\mathrm{He}_d],$$

where He_d is the degree d (probabilists') Hermite polynomial.

Proof. We first restrict ourselves to monomials. As in Theorem 30, we have

$$Đ^n[x^i](x) = \sum_{j=0}^{i} (-1)^{j-i} \frac{n!i!}{(i-j)!j!(n-(i-j))!} x^j.$$

Because we know that the $(1-D)$ operator shifts the average root to the right by 1, it would make sense to cancel this effect if we want to study the asymptotic behavior of the average root gaps. This way, when we normalize to cancel the effect of the gaps' relative growth we might hope for the roots to approach a finite limit. Using the binomial theorem and collecting terms, we get that the x^k coefficient of $Đ^n[x^i](x+n)$ is

$$\sum_{j=0}^{i-k} (-1)^{j-(i-k)} \frac{n!}{(n+j+k-i)!} \binom{i}{j+k} \binom{j+k}{k} n^j$$

The coefficient of n^m in this sum is then

$$(-1)^m \binom{i}{k} \sum_{j=0}^{m} (-1)^j \binom{i-k}{j} \left[\begin{array}{c} i-k-j \\ m-j \end{array} \right].$$

[Rio58] shows that this sum is actually an associated Stirling number of the first kind $d_2(i-k, m)$, so that the coefficient of n^m in the x^k term has the nice form

$$(-1)^m \binom{i}{k} d_2(i-k, m).$$

We can now write

$$Đ^n[x^i](x+n) = \sum_{k=0}^{i} \left[\sum_{m=0}^{i-k} (-1)^m \binom{i}{k} d_2(i-k, m) n^m \right] x^k$$

This can be simplified further. The associated Stirling number of the first kind $d_2(a, b)$ counts the number of permutations of a items that have exactly b cycles, all of which are length at least 2. From this definition it is clear that $d_2(a, b)$ is zero if $b > a/2$. Therefore the index of the inner sum must only go up to $\lfloor \frac{i-k}{2} \rfloor$.

From experiment, we suspect that the asymptotic root gap looks like $\sqrt{n}$. Therefore we should normalize by dividing each root by $\sqrt{n}$. At this point, however, we would first like to generalize the setting back to general polynomials. If we start with a degree d polynomial

$$f(x) = \sum_{i=0}^{d} a_i x^i,$$

then the previous analysis immediately gives

$$Ð^n[f](x+n) = \sum_{i=0}^{d} a_i \left[\sum_{k=0}^{i} \left[\sum_{m=0}^{\lfloor \frac{i-k}{2} \rfloor} (-1)^m \binom{i}{k} d_2(i-k,m) n^m \right] x^k \right]$$

Dividing each root by $\sqrt{n}$, we get

$$\frac{1}{\sqrt{n}^d} Ð^n[f](\sqrt{n}x+n) = \sum_{i=0}^{d} a_i \left[\sum_{k=0}^{i} \left[\sum_{m=0}^{\lfloor \frac{i-k}{2} \rfloor} (-1)^m \binom{i}{k} d_2(i-k,m) \frac{n^m}{n^{(d-k)/2}} \right] x^k \right]$$

If we now let n approach infinity, all terms will go to zero except where $m = \frac{d-k}{2}$. This happens exactly when $i = d$, $i-k$ is even, and $m = \frac{i-k}{2}$, so we get

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}^d} Ð^n[f](\sqrt{n}x+n) = a_d \left[\sum_{\substack{k=0 \\ (d-k) \text{ even}}}^{d} \left[(-1)^{\frac{d-k}{2}} \binom{d}{k} d_2(d-k, \frac{d-k}{2}) \right] x^k \right]$$

Since $d_2(2a,a) = (2a-1)!!$, this gives us

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}^d} Ð^n[f](\sqrt{n}x+n) = a_d \operatorname{He}_d(x),$$

Because the constant a_d does not affect the roots, this gives the result we wanted. □

It is now fairly straightforward to generalize this result to all $(\lambda - D)$ operators. Repeating the first two lines of the proof above, we see that

$$(\lambda - D)^n x^i = \lambda^{n-i} Ð^n[x^i](\lambda x).$$

So if $f(x) = \sum_{i=0}^{d} a_i x^i$ we have

$$(\lambda - D)^n f(x) = \lambda^n \sum_{i=0}^{d} \frac{a_i}{\lambda^i} Ð^n[x^i](\lambda x).$$

At the n-th iteration, then, we have a polynomial which has the same roots as some other polynomial (specifically, the polynomial obtained by applying the multiplier sequence λ^{-i}) after n iterations, except each root is divided by λ. Because the limiting behavior of the root gaps doesn't depend on the initial polynomial, we get

$$G_i\big[(\lambda - D)^n f\big] \sim \sqrt{n} \cdot \frac{G_i[\mathrm{He}_d]}{\lambda}.$$

4. Conclusion

A natural continuation of the project would be to try estimating precisely the lower (resp. upper) bound in Theorem 15, as well to establish new techniques allowing us to give bounds on the movements of some subset of the roots after applying T_λ for some multiplier sequence λ. It would also be interesting to examine how closely the asymptotic result in Theorem 32 approximates the root gap for finite n. A natural extension of this theorem would be to study the application of a sequence of operators $(\lambda_i - D)$. Interestingly, it seems that for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ the normalized root gaps approach finite limits, depending solely on the sequence and the degree of the initial polynomial. In particular, if the sequence consists of a repeated finite sequence $\{\lambda_1, \dots, \lambda_n\}$ the spacings seem to approach the spacings of the corresponding Hermite polynomial divided by the quadratic mean of the λ_i.

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