SUMRY Proposal
Title: Minimizing Closed Geodesics
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1 Part 1

The first part of this project would study the minimizing properties of closed geodesics on doubled regular polygons. These ideas should be immediately accessible to undergraduate students and would provide a jumping off point for the summer.

1.1 basics

A defining property of a geodesic is that it is a locally distance minimizing curve. It is clear that a nontrivial closed geodesic can never be a *globally* distance minimizing curve. Indeed, a closed geodesic cannot minimize past half its length, as traversing the geodesic in the opposite direction always provides a shorter path. It is therefore natural to consider the largest interval on which a given closed geodesic is distance minimizing.

Definition 1.1 ([7], Definition 3.1)**.** *A $1/k$-geodesic is a closed geodesic $\gamma \colon S^1 \to M$ which minimizes on all subintervals of length $l(\gamma)/k$, i.e.*

$$d(\gamma(t), \gamma(t + 2\pi/k)) = l(\gamma)/k \qquad \forall t \in S^1$$

As a first example, we see that the great circles on the sphere are 1/2-geodesics, or half-geodesics. Moreover, by compactness of the circle and local length minimization of geodesics it can be shown that $1/k$-geodesics are as ubiquitous as closed geodesics.

Proposition 1.2 ([7], Theorem 3.1)**.** *Every closed geodesic is a $1/k$-geodesic for some $k \geq 2$.*

1.2 known results

The closed geodesics on the doubled regular polygons are interesting to study because of their simplicity. This project is motivated by the following small result from my thesis work:

Proposition 1.3 ([2], Proposition 2.5)**.** *Let X_n be a doubled regular n-gon.*

1. *If n is odd then X_n has no half-geodesics*
2. *If n is even then X_n has exactly $\frac{n}{2}$ half-geodesics: those curves which pass through the center of each face and perpendicularly through parallel edges.*

Last year at Trinity College I had an undergraduate student work on these ideas and prove the following:

Proposition 1.4 ([4], Theorem 2.5)**.** *Let $\gamma\colon S^1 \to X_n$ be an over-under curve between adjacent edges on a doubled regular n-gon.*

1. *If n is even then γ is a $1/n$-geodesic.*
2. *If n is odd then γ is a $1/2n$-geodesic.*

1.3 open questions

When studying $1/k$-geodesics one often considers the smallest $k \in \mathbb{N}$ for which the space admits a $1/k$-geodesic. To quantify this notion Sormani introduced the minimizing index.

Definition 1.5 ([7], Definition 3.3)**.** *The minimizing index of a space M, denoted minind(M), is the smallest $k \in \mathbb{N}$ such that the space admits a $1/k$-geodesic.*

For n odd the results of the previous section give an upper bound of $2n$ on $minind(X_n)$. Furthermore, we have seen that such X_n do not admit half-geodesics and consequently that $2 < minind(X_n) \leq 2n$. The first question for the SUMRY students would focus on sharpening these bounds on the minimizing index of X_n. For example, given a doubled prime-gon it is compelling to believe that its minimizing index is 2p.

Conjecture 1.6. *If p is an odd prime, then $minind(X_p) = 2p$.*

Observe here that the primality of p is necessary, since if we have that $n = kp$ with $k \geq 2$, we can construct a $1/2p$-geodesic by creating an over under curve between the midpoints of every kth edge of X_n.

Such a result is interesting because it would imply (via techniques in [2]) that the minimizing index of a sequence of smooth manifolds can be unbounded while the Gromov-Hausdorff limit of such a sequence has minimizing index 2.

2 Part 2

The second part of this project would study closed geodesics from a Morse theoretic perspective. The book *Morse Theory* by John Milnor [5] would be great reading for the advanced undergraduate students. This book is self contained and includes a short introduction to Riemannian geometry.

The ideas in the first section are all inspired by a paper by Christina Sormani [7]. This paper introduces the notion of a 1/k-geodesic and describes the basic properties. It goes on to provide a relationship between these curves and the critical points of a discrete Morse energy function. This paper is super accessible (she claims its written at a graduate student level) and includes a number of open problems at the end (most of which are still open). The question which I think might be most interesting and accessible concerns the notion of an index for this discrete energy function.

Problem 11.19: Does the index of the Hessian of the uniform energy provide an estimate on the minimizing index? It would be interesting to investigate whether the minimizing index of an openly 1/k-geodesic is related to the Hessian of the uniform energy.

The above question essentially asks us to generalize the second derivative test to this setting. This should be accessible to a motivated undergraduate. There are a number of other questions posed by Sormani that I think are interesting and accessible. Here are a few:

Problem 11.15: What conditions can be placed on a manifold or metric space to prove the existence of a critical point of the uniform energy?

Problem 11.8: What properties can be placed on a simply connected manifold to guarantee the existence of a half-geodesic.

3 Part 3

There are a few other related questions that might be interesting to consider.

Question 1: An question that incorporates the above ideas is the classification of flat tori by their length spectra. It is known that pairs of flat tori are Laplace isospectral iff they are length isospectral. One can then ask if the length spectrum determines the isometry class of the torus. This is an easy exercise in dimension 2 and a hard proof in dimension 3. Counterexamples exist in dimensions 4 and higher.

It is not hard to show that the half-geodesic length spectrum determines the isometry class for flat two tori. Extensions of this result to dimensions higher than 2 seem reasonable. If more information is needed in higher dimensions, one can ask if some subset of the 1/k-geodesic length spectrum determines the torus.

Question 2: Sormani and Shankar [6] provide a notion of *conjugate point* for the length space setting. It would be interesting to explore the relationship between this notion of conjugate point and the generalized notion of critical point for the uniform energy introduced in [1]. See [7] Problem 11.5, Remark 10.6, and Section 6.

Question 3: One could easily ask the same questions about the Platonic solids that we have asked about doubled polygons. In particular, do these surfaces admit half-geodesics, and if not, what is the minimizing index of each solid? I saw a senior thesis at Dartmouth that considered *hermit points* on the octahedron, work that could potentially be used to study the existence of half-geodesics.

References

[1] I. Adelstein. Minimizing closed geodesics via critical points of the uniform energy, *Math. Res. Lett.*, **23** (2016), 953-972.

[2] I. Adelstein. Existence and non-existence of half-geodesics on S^2, *Proc. Amer. Math. Soc.*, **144** (2016), 3085-3091.

[3] I. Adelstein and J. Epstein. Morse theory for the uniform energy, *J. Geom.*, to appear, 1609.09357.

[4] I. Adelstein and A. Fong. Shortest paths on doubled polygons, preprint.

[5] J. Milnor. *Morse Theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[6] K. Shankar and C. Sormani, Conjugate points in length space, Adv. Math. 220(3): 791-830, 2009.

[7] C. Sormani, Convergence and the length spectrum, Adv. Math. 213(1): 405439, 2007.