Intersecting graph families (SUMRY 2018 project description)

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High-level description: This project explores a problem in extremal graph theory related to the Erdős–Ko–Rado theorem [5]. Participants will learn about and combine ideas from graph theory and probabilistic combinatorics (as well as computer-generated output) to find (or disprove the existence of) constructions of large graph families having certain intersection properties. This project requires relatively little background knowledge to get involved, although applicants with sufficiently strong backgrounds (e.g., those taking Ross's modern combinatorics class on Fourier analysis) will have plenty of room to explore a variety of advanced techniques.

Background and motivation

One of the most important results in extremal combinatorics is the Erdős–Ko–Rado theorem [5], which states the following:

Theorem 1 (Erdős–Ko–Rado [5], 1961)**.** *Suppose $n \geq 2k$, and $\mathcal{F}$ is a collection of k-element subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. (Moreover for $n > 2k$, equality is attained iff $\mathcal{F}$ is a collection of all k-element sets containing some particular element.)*

This has been refined and generalized in many ways (e.g., the equally classic Hilton-Milner theorem [7]), and it has spawned a host of related questions (see [6] for a survey) including the following, which is our main focus

Main question: Let H be a fixed (nonempty) graph, and suppose $\mathcal{F}$ is a family of graphs on a common vertex set V such that $A \cap B$ contains a copy of H for all $A, B \in \mathcal{F}$. How large can $|\mathcal{F}|$ be?

For notational convenience, consider $\mu(\mathcal{F}) = |\mathcal{F}|/2^{|V|(|V|-1)/2}$, which is the fraction of graphs on V in $\mathcal{F}$ (with this normalization, $0 \leq \mu(\mathcal{F}) \leq 1$). Henceforth, $\mathcal{F}$ will be assumed to satisfy the above intersection hypothesis.

Brief overview of what's known

As we will discuss in subsequent sections (but which you should pause and try to prove for yourself), we always have $\mu(\mathcal{F}) \leq 1/2$, whereas there are collections with $\mu(\mathcal{F}) = 2^{-e(H)}$, where $e(H)$ is the number of edges of H.

An old conjecture of Simonovits and Sós says that if H is a triangle, then $\mu(H) \leq 1/8$. The first progress on this was in 1986 by Chung, Graham, Frankl, and Shearer [3], who used an entropy argument to prove that if H is not bipartite[1] then $\mu(\mathcal{F}) \leq 1/4$. Decades later Ellis, Filmus, and Friedgut [4] made a groundbreaking improvement of $\mu(\mathcal{F}) \leq 1/8$ for non-bipartite H (completely settling the triangle case).

For bipartite H, much less is known. If H is a disjoint union of stars,[2] there are families with[3] $\mu(\mathcal{F}) = 1/2 - o(1)$, and in [2] it was shown that if H is the 3-edge path, there are families with $\mu(\mathcal{F}) > 2^{-e(H)}$. *But that's all we know!* In particular, we do not know of any fixed bipartite H for which $\mu(\mathcal{F})$ is bounded away from 1/2, although Alon [1] conjectures that if H is not the disjoint union of stars, this is always the case.

More in-depth discussion of existing proofs

First, it is easy to see that $\mu(\mathcal{F}) \leq 1/2$ simply because any two graphs in $\mathcal{F}$ have non-empty intersection (implying $\mathcal{F}$ cannot contain both a graph and its complement). And for a family with $\mu(\mathcal{F}) = 2^{-e(H)}$, just let $\mathcal{F}$ be all graphs containing *some particular copy* of H (as in Erdős–Ko–Rado).

The entropy proof of [3] is the first application of a result known as *Shearer's lemma*. Entropy is a beautiful, elementary, and shockingly useful concept that I don't expect you to currently know about. Instead of trying to learn from [3], a vastly better resource would be [8], which is a lovely introduction to the topic (including the proof from [3] as an early example).

Ellis, Filmus, and Friedgut [4] approach the problem via more sophisticated and nuanced techniques from Fourier analysis and spectral methods, which are beyond the scope of (and likely irrelevant to) this project.

As for bipartite H, the result in [2] is simply a small example, and to get a taste for star forests, suppose for simplicity that H is a single star with k edges. Pick some $x \in V$, and let $\mathcal{F}$ consist of all graphs where the degree of x is at least $(|V| + k)/2$. The intersection of any two such graphs has x with degree at least k (thus, it contains a copy of H centered at x), and $\mu(\mathcal{F}) = 1/2 - o(1)$ by (say) the central limit theorem[4] [since $k = o(\sqrt{|V|})$].

[1] A graph is *not bipartite* iff it contains a cycle of odd length (a triangle, a pentagon, et cetera). Equivalently, a graph is not bipartite iff its *chromatic number* is at least 3.

[2] A *star* is a connected graph with only one vertex of degree more than 1.

[3] Following standard notation, $o(1)$ represents a quantity that tends to 0 as the relevant parameter (here $|V|$) tends to infinity.

[4] Here, we use the fact that if G is chosen uniformly at random from all graphs on V, then the degree of x is a symmetric binomial random variable with mean $(|V| - 1)/2$. This is a good example of how probability is likely to show up in this project.

More details on SUMRY project

Requirements of applicants

Here are things that would be good to know coming into the project. I *do not* expect applicants to know all of these things [especially not 4], but if you don't care to learn about these topics, then you won't like the project.

1. It would be good for applicants to have seen some discrete math especially graph theory, but those concepts are easy to pick up.

2. This will likely involve more probability than you think (but not a ton). While some background would be very helpful (e.g., random variables, linearity of expectation, central limit theorem), a lot can be learned "on the job" so don't worry.

3. Some programming experience would be useful for exploring small cases, testing conjectures, and looking for examples.

4. (Optional, but great) Those taking Ross's spring 2018 modern combinatorics class on discrete Fourier analysis would be *particularly* well prepared, and that would enable us to pursue ideas along the lines of [4]. If you are interested in this project, and you are taking Ross's class, please note this in your application. (This is *certainly not* a requirement, but if there are enough folks with this background it could open a new direction for the project.)

What participants will learn

In addition to the current literature on the subject [3, 4, 2], participants will learn about (and practice) the following

- probability (moment estimates, central limit theorem, et cetera)
- entropy and information theory (e.g., participants will read [8])
- extremal graph theory
- programming (at least *somebody* will be coding for sure)
- (optional) random graphs
- (optional) Fourier analysis

A few more hints about the project

We will focus on some combination of these ideas (among others) following student interest as well as what seems fruitful.

- what can be said for small fixed H? (especially for trees)

- what about some huge bipartite graph like $H = K_{1000,1000}$?
- what can be said if H is the union or product of smaller graphs?
- what if we restrict $\mathcal{F}$ to be subgraphs of some Γ? (generalizing the original version of the question where $\Gamma = K_n$, the complete graph)

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, John Wiley & Sons, 2004.

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[3] F. Chung, R. Graham, P. Frankl, and J. Shearer, *Some intersection theorems for ordered sets and graphs*, Journal of Combinatorial Theory, Series A, 43 (1986), pp. 23–37.

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