PROPOSAL FOR SUMRY 2018
BLASCHKE UNWINDING

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Contents

1. The Theoretical Problem

Let $p_0 : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree n. We consider the following algorithm of obtaining a sequence of polynomials.

(1) Given a polynomial p_k, set $q_k(z) = p_k(z) - p_k(0)$.
(2) Then find all roots $z_0, \dots, z_\ell$ of q_k inside the unit disk $\mathbb{D}$ and define

$$p_{k+1}(z) = q_k(z) \prod_{j=1}^{\ell} \frac{1 - \overline{z_j} z}{z - z_j}$$

and go to (1).

This produces a sequence of polynomials. We observe that (1) implies that 0 is always a root and thus

$$\deg p_{k+1} \leq \deg p_k - 1.$$

Questions.

(1) How does this sequence of polynomials behave?
(2) It is guaranteed to reach constants after at most n steps but experiments suggests that this happens much, much faster. What bounds can be proven?
(3) Does the process have a 'unifying' structure? Does p_5 tend to look a certain way that is not strongly depending on p_0?
(4) The study of polynomials on the complex plane has many seminal contributions; entire books (see e.g. Borwein and Erdelyi [1]) are written about certain inequalities relating them to their roots. Can this be used?
(5) What can be said about this process if p_0 is a holomorphic function?

2. Background

This process, applied to general holomorphic functions $p_0 : \mathbb{C} \rightarrow \mathbb{C}$, was first invented by Raphy Coifman in the 1990s. It was the subject of a PhD thesis (Nahon [5], PhD Yale, 2001) where it was experimentally established that this process, that can be reinterpreted as a nonlinear analogue of Fourier series, has a series of remarkable properties (among other things, it was used in speech recognition). In particular, which is nice in numerical applications, the process can be carried out without ever computing the roots of the functions: this is an idea dating back to a paper of Guido and Mary Weiss from 1962 [10]. However, theoretical results remained elusive. In the meanwhile, Tao Qian from the University of Macau developed and studied the same process independently [6, 7, 8, 9] and proved a basic convergence result.

Recently, Coifman & Steinerberger [2] showed that the process is incredibly rich in mathematical structure and serves as a contraction at every level of regularity (see [2]) for details. A follow-up paper [3], joint with Hau-tieng Wu, showed that the process is indeed useful in practice and allows to obtain fast rational approximation of functions. Many theoretical problems remain. Perhaps the most tempting question is the speed of convergence, which seems to be exponential in numerical experiments: this is related to the behavior of the roots in the complex plane when interpreted as a dynamical system. Studying polynomials seems a natural first step (especially since, by density considerations, polynomials should exhibit generic behavior under the process and most proofs are likely to translate to the more general setting of general holomorphic functions).

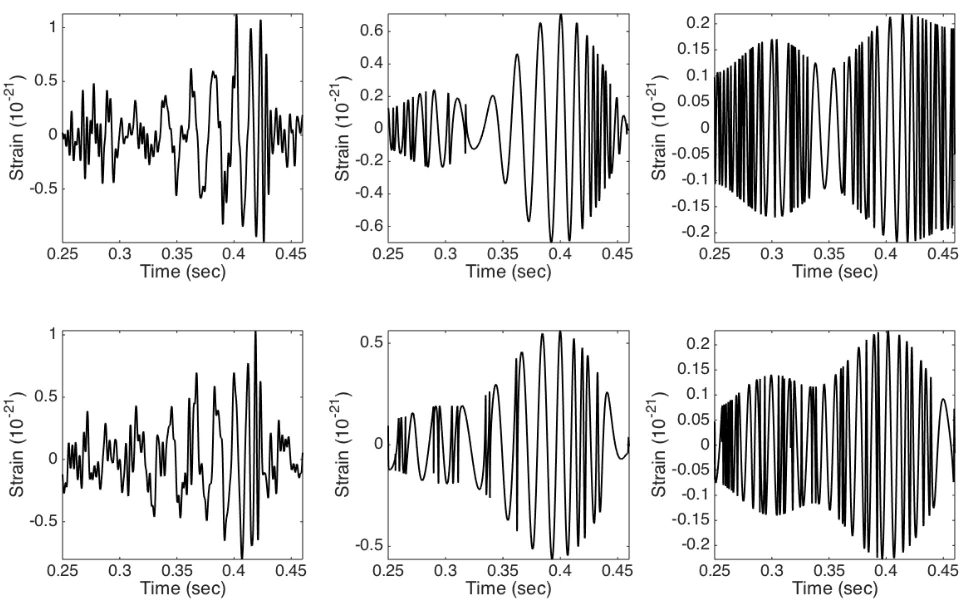

FIGURE 1. An expansion of a gravity wave ever detected: the signal (left), the first Blaschke product of the respective signal (middle) and the second Blaschke product (right).

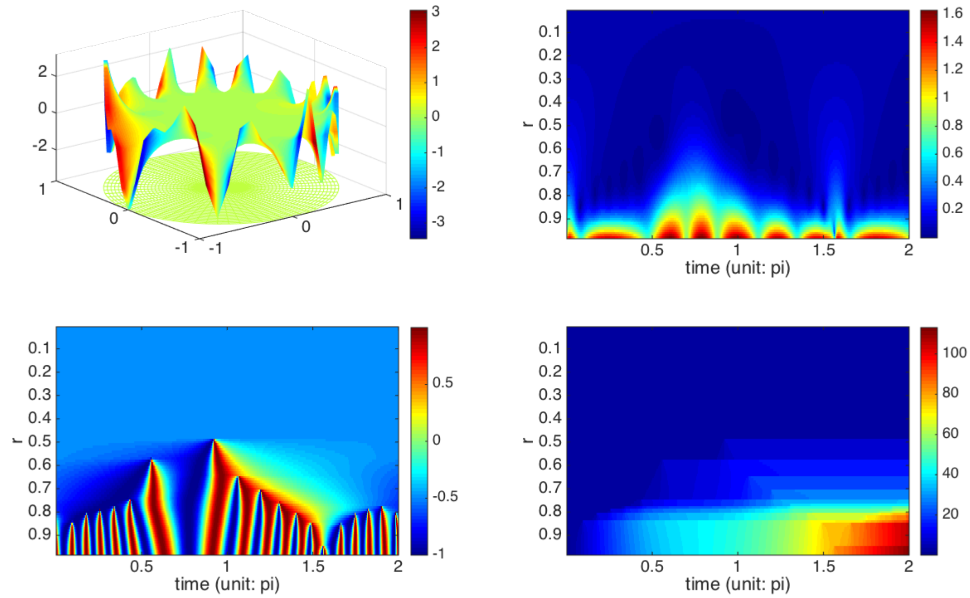

FIGURE 2. A function f (upper left), its absolute value written in polar coordinates (upper right), changes in the phase pointing towards the location of the roots (lower left) and the total phase in polar coordinates (lower right).

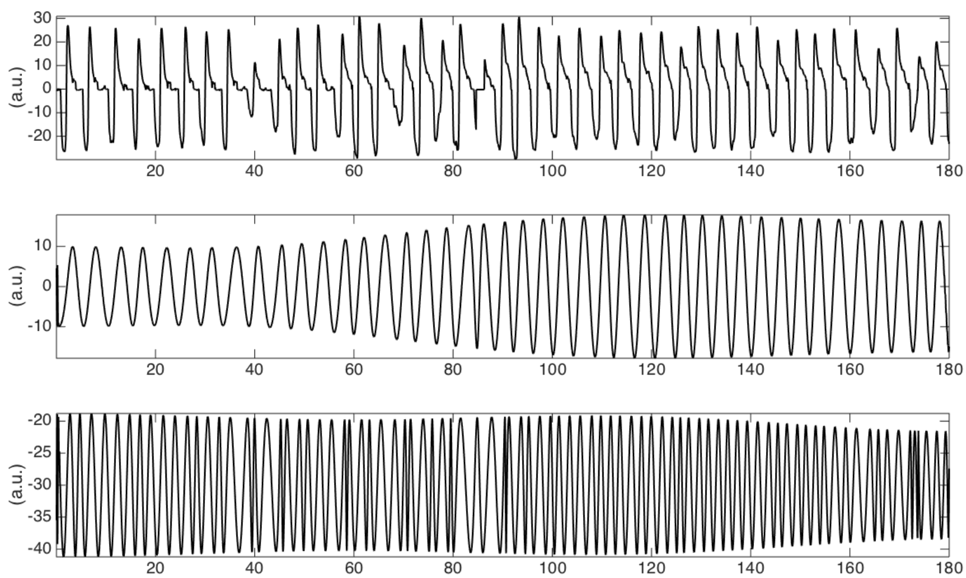

FIGURE 3. An expansion of a respiratory signal (from [3]): subtle changes in the frequency are difficult to detect by classical means but carry a lot of medically relevant information.

3. The Practical Problem

The second aspect of the problem is more practical and greatly benefits from the medical expertise of the second mentor (who has a medical degree and works actively on mathematical aspects of medical signal processing). A truly fundamental problem is the following

> **Problem.** Given a signal comprised of multiple components (for example the heartbeats of both a pregnant woman and the fetus), is it possible to decompose them into separate signals?

Can this iterative polynomial unwinding scheme be used in combination with other techniques, for example diffusion maps, random projections, Fourier-series based methods etc.? We have numerical evidence that it improves the performance of the synchrosqueezing transform [4] (invented by Daubechies, Lu and the second mentor), which is the current standard in the field for frequency extraction. The hope is that information acquired from this decomposition can be used to study general time series from a more geometric viewpoint.

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