Large gaps between zeros of GL(2) L-functions

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Outline

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The Riemann Zeta-Function

The Riemann zeta-function $\zeta(s)$ is given for $\Re(s) > 1$ by the following absolutely convergent Dirichlet series and Euler product:

$$\zeta(s) = \sum_{n} n^{-s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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Riemann Hypothesis: All non-trivial zeros have $\Re(s) = \frac{1}{2}$.

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- The Euler product of $\zeta(s)$ translates knowledge about zeros of $\zeta(s)$ to knowledge about the distribution of prime numbers.
- Other classes of L-functions encode information about many mathematical objects, e.g., ranks of elliptic curves and class numbers of imaginary quadratic fields.

Spacings between Zeros

- Classical question: how are the spacings between consecutive critical zeros distributed?
- Numerical observation: spacings between zeros behave statistically similarly to spacings between eigenvalues of large complex Hermitian matrices.

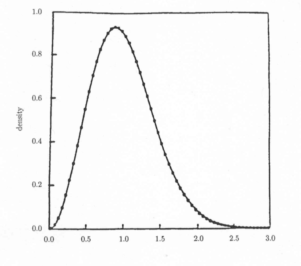

Figure: Critical zeros of $\zeta(s)$ and Hermitian matrix eigenvalues.

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$$\text{Letting} \quad \Lambda = \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}},$$

this conjecture is equivalent to $\lambda = \infty$. Few nontrivial results have been established. Even for the Riemann zeta function, unconditionally it is only known that

$$\Lambda > 2.69.$$

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Types of L-functions

- One way that we characterize L-functions depends on a notion of *degree*.
- An analytic way to characterize degree is to look at the order in p^{-s} of the local $\mathcal{L}_p$ factors in the Euler product.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \leftarrow \text{degree } 1$$

$$L(s, f) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \leftarrow \text{degree } 2$$

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- A *primitive* L-function is one that cannot be written as the product of two L-functions. (For example, the Dedekind zeta function for a quadratic number field K is not primitive because it factors as $\zeta_K(s) = \zeta(s)L(s, \chi_d)$.)

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- More generally, let χ be a Dirichlet character. We then form the Dirichlet L-function $L(s, \chi)$ for $\Re(s) > 1$ by the following absolutely convergent Dirichlet series and Euler product:

$$L(s, \chi) = \sum_{n \geqslant 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

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$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz).$$

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These normalized Fourier coefficients $\lambda_f(n)$ are of arithmetic interest.
- Form the L-function associated to f from the absolutely convergent Dirichlet series with the $\lambda_f(n)$ as coefficients:
$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \qquad (\Re(s) > 1).$$

A primitive degree 2 L-function

- $L(s, f)$ admits an Euler product of degree 2:

$$L(s,f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}$$

(where $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$).

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- $L(s, f)$ is primitive.

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Statement of results

We proved the following unconditional theorem:

Theorem 1 (BMMRTW '14).
Let $\{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let $\kappa_T = \max\{\gamma_{n+1} - \gamma_n : T + 1 \leqslant \gamma_n \leqslant 2T - 1\}$. Then

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right),$$

where c_f is a constant that encodes arithmetic information.

Statement of results

If we assume GRH for interpretive purposes, this means there are infinitely many normalized gaps between consecutive zeroes that are at least $\sqrt{3}$ times the mean spacing.

Shifted Moment Result

In order to prove our theorem, we use a method due to R.R. Hall, along with the following shifted moment result:

Theorem 2 (BMMRTW '14).

$$\int_T^{2T} L\left(\frac{1}{2}+it+\alpha, f\right) L\left(\frac{1}{2}-it+\beta, f\right) dt$$
$$= c_f T \sum_{n\geqslant 0} \frac{(-1)^n 2^{n+1}(\alpha+\beta)^n (\log T)^{n+1}}{(n+1)!} + O(T(\log T)^{1-\delta}),$$

where $\alpha, \beta \in \mathbf{C}$ and $\alpha, \beta \ll 1/\log T$.

Shifted Moments Proof Technique

- Following a method due to Ramachandra, we consider

$$L(s+\alpha,f)=\sum_{n\geqslant 1}\frac{\lambda_f(n)}{n^{s+\alpha}}e^{-\frac{n}{X}}+F(s)\sum_{n\leqslant X}\frac{\lambda_f(n)}{n^{1-s-\alpha}}+E(s),$$

where $\lambda_f(n)$ are the Fourier coefficients of $L(s,f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.

Shifted Moments Proof Technique

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$$L(s+\alpha, f) = \sum_{n \geqslant 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n \leqslant X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s),$$

  where $\lambda_f(n)$ are the Fourier coefficients of $L(s, f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.
- We have an analogous expression for $L(1-s+\beta, f)$

Shifted Moments Proof Technique

- We consider the product

$$L(s+\alpha, f)L(1-s+\beta, f).$$

where each factor gives rise to four products, resulting in sixteen total products to estimate.

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where each factor gives rise to four products, resulting in sixteen total products to estimate.
- Using a generalization of Montgomery and Vaughan's mean value theorem and contour integration we are able to estimate this product and compute the resulting moments.

Shifted Moment Result for Derivatives

- The shifted moment result allows us to deduce lower order terms and moments of derivatives of L-functions by means of differentiation and the Cauchy Integral Formula.

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- The shifted moment result allows us to deduce lower order terms and moments of derivatives of L-functions by means of differentiation and the Cauchy Integral Formula.
- We derive an expression for

$$\int_{T}^{2T} L^{(\mu)}\left(\frac{1}{2}+it, f\right) L^{(\nu)}\left(\frac{1}{2}-it, f\right)\, dt,$$

where $T \geqslant 2$ and $\mu, \nu \in \mathbf{Z}^{+}$. We use this result in Hall's method to obtain the lower bound stated in our Theorem.

Modified Wirtinger Inequality

Using Hall's method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg:

Lemma 1 (Bredberg).
Let $y : [a, b] \to \mathbf{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$\int_a^b |y(x)|^2 dx \leqslant \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 dx.$$

Proving our Result

- We define the function

$$g(t) := \left(e^{i\rho t \log T}\right) L\left(\frac{1}{2} + it, f\right).$$

We fix f and let $\tilde{\gamma}$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.

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We fix f and let $\tilde{\gamma}$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.
- The crucial property of this function is that it has the same zeroes as $L(s, f)$, namely $g(t) = 0$ when $t = \tilde{\gamma}$. We use this function in the modified Wirtinger's inequality.

Proving our Result

- We apply sub-convexity bounds along the critical line to establish:

$$\int_T^{2T} |g(t)|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 dt + O\left(T^{\frac{2}{3}} (\log T)^{\frac{5}{6}}\right).$$

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- Noting that our $g(t)$ and $g'(t)$ may be expressed in terms of $L\left(\frac{1}{2} + it, f\right)$, we can write our inequality explicitly in terms of formulæ given by our theorem(s) for moments of L-functions.

Finishing the Proof

- After substituting our formulæ, we have the inequality:

$$\frac{\kappa_T^2}{\pi^2} \geqslant \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left(1 + O(\log T)^{-\delta}\right).$$

Finishing the Proof

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$$\frac{\kappa_T^2}{\pi^2} \geqslant \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left(1 + O(\log T)^{-\delta}\right).$$

- We are able to minimize this by setting $\rho = 1$, so we have our desired result

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T} \left(1 + O\left(\frac{1}{c_f} (\log T)^{-\delta}\right)\right).$$

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