Strategy in All-Pay Bidding Games
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Introduction
In all-pay auctions, only one bidder wins but all bidders must pay the auctioneer. All-pay bidding games arise from attaching a similar bidding structure to traditional combinatorial games to determine which player moves next. In contrast to the established theory of single-pay bidding games, optimal play involves bidding from a probability distribution.

Our work has focused on better understanding the traditional combinatorial games to determine which strategy is called a grounded strategy. Both players play strategies such that neither player can compute a strategy from that matrix.

Definitions
Definition. A payoff $v_A$ for player $A$ in $G_{a,b}$ is equal to the probability that player $A$ wins the game under optimal play. A payoff matrix $M_A$ for player $A$ is given by $(M_A)_{i,j} = v_A(G_{a,i+j,b-j+1})$ where $G'$ is the game position after the next move. We let $M_A(k)$ denote the $k \times k$ principal minor of the payoff matrix.

Definition. A strategy $S_A(G_{a,b})$ for player $A$ is given by an $(a+1)$-dimensional probability vector where $S_A(i)$ gives the probability that player $A$ will bid $i$ chips. A strategy $S$ is gap-free if $S_i, S_j \neq 0 \iff S_k \neq 0 \forall i \leq k \leq j$. A strategy $S$ has length $|S| = l(S) \iff S_{l-1} \neq 0$ and $S_m = 0 \forall m \geq 1$.

Definition. $G$ is called precise if in every successor state to $G$, it is strictly better to have one more chip.

Definition. A game is in Nash equilibrium if both players play strategies such that neither player can further improve his or her payoff.

Definition A strategy is called a Nash equilibrium strategy if it guarantees a player the payoff he would recieve in a Nash equilibrium. Von Neumann and Morgenstern proved that in two-player zero-sum games a player's Nash equilibrium strategies are interchangeable [1].

Our Results

Lemma 1. For a Nash equilibrium with opposing strategies $S_A$ and $S_B$ if $(S_A)_i \neq 0$ then $(M_A S_B)_i = v_B$.

Assume, without loss of generality, that player $A$ has the tie-breaking advantage at a given turn. Let the game in question, $G_{a,b}$, be precise.

Lemma 2. Player $A$ has a gap-free and grounded optimal strategy. Player $B$ has a gap-free optimal strategy.

Definition. The reverse of a length $l$ strategy $S$ is given by $R(S) = R((s_0,s_2,s_4,\ldots,s_{l-1},0,\ldots,0)) = (s_1,s_3,s_5,\ldots,s_{l-2},0,\ldots,0)$, where the number of trailing zeroes will be clear from context.

Reverse Theorem
Let $S$ be a Nash equilibrium strategy for player $A$. Then $R(S)$ is a Nash equilibrium strategy for his opponent.

Corollary. Player $A$ and player $B$ have optimal strategies of the same length.

Lemma 3. Let the length of $A$'s Nash equilibrium strategy of maximum length be $l$. Then there exists a valid strategy for player $A$ that produces the same payoff against player $B$'s first $k$ pure strategies if and only if $k \leq l$.

Our Algorithm
By Lemmas 1 and 2, if $S_A$ has length $l$, we have $M_A(l) \cdot S_A = v_A \cdot 1_l$. Thus, if we know what $l$ is, we can compute $S_A$ by evaluating $M_A(l)^{-1} \cdot 1_l$ and normalizing the result into a probability vector.

Thus, the main problem is to find the length of the equilibrium strategy. Using Lemma 3 above, we can employ a binary search to find the length of a player's strategy in $O(\log(n) \cdot n^2)$ time, where $n$ is the player's chip count.

A Sample Bidding Game
Consider the following “Best of Three” game in which the first player to make two moves wins. Each player begins with 100 chips.

<table>
<thead>
<tr>
<th>Turn</th>
<th>A</th>
<th>B</th>
<th>A's bid</th>
<th>B's bid</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>50</td>
<td>0-1</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>70</td>
<td>45</td>
<td>30</td>
<td>1-1</td>
</tr>
<tr>
<td>3</td>
<td>115</td>
<td>85</td>
<td>115</td>
<td>85</td>
<td>2-1</td>
</tr>
</tbody>
</table>

Note that updating each player’s chip count only depends on the difference between their bids, not the bid values themselves. In the event of tied bids, the player with more chips gets to move. One player is arbitrarily designated to win ties when chip counts are equal.

Formalizing Bidding Games
We represent games as directed graphs, where vertices are game states and edges are possible moves. The below graph represents the “Best of Three” game from the example above.

References