Some Results on 2-Lifts of Graphs

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1 Ramanujan Graphs

Let G be a d-regular graph. Every d-regular graph has d as an eigenvalue (with multiplicity equal to the number of connected components), and all other eigenvalues have magnitude less that or equal to d ($-d$ is an eigenvalue if and only if G is bipartite). Eigenvalues of magnitude d are called **trivial eigenvalues**. Let $\lambda(G)$ be defined in the following way:

$$\lambda(G) = \max_{|\lambda_i|<d} \mid \lambda_i \mid$$

Thus $\lambda(G)$ is the greatest magnitude of the non-trivial eigenvalues of a regular graph. A graph is called a **Ramanujan graph** if $\lambda(G) \leq 2\sqrt{d-1}$. Ramanujan graphs represent an extremal class of graphs as made evident by the following theorem by Alon and Boppana [1]:

Theorem 1.1. *For every integer $d \geq 3$ and every $\epsilon > 0$, there exists an n_0 so that every d-regular graph G with more than n_0 vertices has a non-trivial eigenvalue that is greater than $2\sqrt{d-1} - \epsilon$.*

2 Signings and 2-Lifts

A **signing** of a graph is a function $s : E(G) \mapsto \{-1, 1\}$. Signings prescribe a way of performing **2-lifts** of graphs in the following way: create two copies of the vertex set of G, $V_1(G)$ and $V_2(G)$. For each $e = a \sim b \in E(G)$, if e is positively signed, then create edges $a_1 \sim b_1$ and $a_2 \sim b_2$, and if e is negatively signed, then create edges $a_1 \sim b_2$ and $a_2 \sim b_1$. Denote the resulting graph as G_s.

Let A_s^+ (A_s^-) denote the (adjacency matrix of the) subgraph of G containing only the positively (negatively) signed edges. Notice that $A_s^+ + A_s^- = A$. Let $A_s = A_s^+ - A_s^-$. We call A_s the **signed adjacency matrix**.

Proposition 2.1 ([2])**.** $\mathcal{Spec}(G_s) = \mathcal{Spec}(G) \cup \mathcal{Spec}(A_s)$

Notice then that if we start with a d-regular graph G that is Ramanujan, and we are able to find a signing s such that $\rho(A_s) \leq 2\sqrt{d-1}$ (ρ here is the spectral radius), then the resulting graph will also be Ramanujan. If we are able to repeat this process at each step, then we would obtain an infinite family of Ramanujan graphs.

3 Paley Graphs

Definition 3.1. *Let $n = p^k \equiv 1 \mod 4$ with p prime. The **Paley graph** on n vertices (P_n) is defined in the following way: let F_n be the finite field on n elements. Then, $V(P_n) = F_n$, and $E(P_n) = \{(a,b) \in F_n \times F_n : a-b \in (F_n^\times)^2\}$.*

Definition 3.2. *Let H be a graph on n vertices. The **complement** of H, denoted $\overline{H}$ or $K_n \backslash H$, has $V = V(H)$ and $E = \{(a,b) \in V \times V : (a,b) \notin E(H)\}$.*

Paley graphs exhibit a number of nice properties:

Proposition 3.3. *Let $n = p^k \equiv 1 \mod 4$. Then*

1. *$\mathcal{Spec}(P_n) = \{\frac{1}{2}(n-1) : 1, \frac{1}{2}(-1+\sqrt{n}) : \frac{1}{2}(n-1), \frac{1}{2}(-1-\sqrt{n}) : \frac{1}{2}(n-1)\}$ with $\vec{1}$ as the eigenvector corresponding to the eigenvalue $\frac{1}{2}(n-1)$ (strong regularity).*

2. *$P_n \cong \overline{P_n}$ (self-complementarity).*

Paley graphs give us a way of partitioning the complete graph on n vertices (K_n) into two isomorphic pieces.

Proposition 3.4. *Let $n = p^k \equiv 1 \mod 4$, and $G = K_n$. Let $A_s^+ = P_n$ for some $P_n \subset K_n$ and $A_s^- = \overline{P_n}$. Then $\mathcal{Spec}(A_s) = \{0 : 1, \sqrt{n} : \frac{1}{2}(n-1), -\sqrt{n} : \frac{1}{2}(n-1)\}$.*

Proof. Recall that $\mathcal{Spec}(K_n) = \{n-1 : 1, -1 : n-1\}$ and that $\vec{1}$ is the eigenvector associated with the eigenvalue $n-1$. Notice that we can write $A_s = 2A_s^+ - K_n$. Since $\vec{1}$ is an eigenvector of both A_s^+ and K_n, we know it is also an eigenvector of A_s with corresponding eigenvalue $2 * \frac{1}{2}(n-1) - (n-$

$1) = 0$. By the spectral theorem, we know that all other eigenvectors of A_s^+ are orthogonal to $\vec{1}$, as are all other eigenvectors of K_n, all of which have eigenvalues of -1. Thus, all eigenvectors of A_s^+ are eigenvectors of A_s. Each eigenvalue of A_s^+ of value $\frac{1}{2}(-1+\sqrt{n})$ becomes an eigenvalue of A_s of value $2 * \frac{1}{2}(-1+\sqrt{n})+1 = \sqrt{n}$ (and similarly the eigenvalues $\frac{1}{2}(-1-\sqrt{n})$ become $-\sqrt{n}$).

□

Since $\sqrt{n} \leq 2\sqrt{d-1} = 2\sqrt{n-2}$ for all $n \geq 3$, 2-lifts of K_n performed in this way are Ramanujan. However, this only works when there is a Paley graph on n vertices. We claim that it is in fact possible to perform a Ramanujan 2-lift for all complete graphs.

Theorem 3.5. $\forall\ 5 \leq \ell \in \mathbb{N}$, $\exists\ n = p^k \equiv 1 \mod 4 \geq \ell$ *such that every signing s derived from a vertex set $\sigma \subseteq V(K_n)$ of size ℓ with $A_s^+ = (P_n)_\sigma$ and $A_s^- = (\overline{P_n})_\sigma$ yields a Ramanujan 2-lift.*

Here $(P_n)_\sigma$ refers to the vertex-induced subgraph of P_n on the vertices in σ. In order to prove this theorem, we first need the following two lemmas:

Lemma 3.6 ([4])**.** *(Cauchy's interlacing theorem) If A is a Hermitian matrix, and B is a principal submatrix of A, then the eigenvalues of B interlace the eigenvalues of A.*

What's particularly notebale about this theorem in this case is that when we have both positive and negative eigenvalues, $\rho(B) \leq \rho(A)$ for all principal submatrices B of A.

Lemma 3.7 ([9])**.** $\forall\ 6 \leq \ell \in \mathbb{N}$, $\exists$ *at least two primes between ℓ and 2ℓ.*

Proof. (Theorem 3.5) Let $n = p^k \equiv 1 \mod 4$. Then by Proposition 3.4, we can partition K_n into two isomorphic pieces and perform a 2-lift via the signing s with $A_s^+(K_n) = P_n$ and $A_s^-(K_n) = \overline{P_n}$. If we choose a subset σ of ℓ of the vertices of K_n, then we can derive a signing s^* for K_ℓ with $A_{s^*}^+(K_\ell) = (P_n)_\sigma$ and $A_{s^*}^-(K_\ell) = (\overline{P_n})_\sigma$. Notice then that A_{s^*} is a principal submatrix of A_s, so $\rho(A_{s^*}) \leq \rho(A_s) = \sqrt{n}$ by Lemma 3.6. Thus, if $\sqrt{n} \leq 2\sqrt{\ell-2}$, then s^* necessarily yields a Ramanujan 2-lift of K_ℓ. If we are given ℓ, and we are able to find an $n \geq \ell$ for which a Paley graph on n vertices exists such that $\sqrt{n} \leq 2\sqrt{\ell-2}$ (or equivalently $\ell \leq n \leq 4n-8$), then following the procedure just described will produce a Ramanujan 2-lift of K_ℓ.

In order to apply Lemma 3.7 in the following proof, we need $\lceil\sqrt{\ell}\rceil \geq 6$, which implies $\ell \geq 26$. It can be easily checked by hand that for all $5 \leq \ell \leq 25$, there is an n sufficiently close by for the prescribed procedure to produce a

Ramanujan 2-lift. Now assume that $\ell \geq 26$. Notice that for all primes $p \neq 2, p^2 \equiv 1 \mod 4$, so it would be sufficient to find $n = p^2$ such that $\ell \leq p^2 \leq 4n - 8$, or equivalently $\sqrt{\ell} \leq p \leq 2\sqrt{\ell - 2}$. By Lemma 3.7, there exist at least two primes between $\lceil\sqrt{\ell}\rceil$ and $2\lceil\sqrt{\ell}\rceil$. Since $2\lceil\sqrt{\ell}\rceil$ is even, it is not prime, so the largest the largest prime in this range can be is $2\lceil\sqrt{\ell}\rceil - 1$ and the largest the second largest prime in this range can be is $2\lceil\sqrt{\ell}\rceil - 3$. We get the following inequality:

$$\sqrt{\ell} \leq \lceil\sqrt{\ell}\rceil \leq p \leq 2\lceil\sqrt{\ell}\rceil - 3 < 2\sqrt{\ell} + 2 - 3 = 2\sqrt{\ell} - 1 \leq 2\sqrt{\ell - 2}$$

Thus there is necessarily an $n = p^2$ sufficiently close by for all $\ell \geq 26$. □

4 Matchings

Definition 4.1. *A **k-matching** in a graph is a collection of k edges such that no two edges in the set share a vertex. We denote the number of k-matchings in a graph by m_k.*

Definition 4.2. *The **matching polynomial** of a graph G is:*

$$\mu_G(x) := \sum_{k \geq 0} (-1)^k m_k x^{n-2k}$$

Notice that this is always an even or odd function, so the zeros of $\mu_G(x)$ are symmetrically distributed. The matching polynomial is a very well-studied object and has a number of nice properties:

Theorem 4.3 ([6])**.** *Let G be a graph. Then,*

1. *$\mu_G(x)$ is real-rooted.*
2. *The roots of $\mu_G(s)$ all lie in the range $[-2\sqrt{\Delta(G) - 1}, 2\sqrt{\Delta(G) - 1}]$, where $\Delta(G)$ is the max degree of all of the vertices in G.*

Notice that in the case where G is a d-regular graph, all of the roots have magnitude less than or equal to $2\sqrt{d-1}$, which is exactly the Ramanujan bound. Indeed, there is a strong connection between the matching polynomial and signings of a graph:

Theorem 4.4 ([5])**.** *Let G be a graph and let S be the set of all possible signings s on G. Then,*

$$\mathbb{E}\ det(xI - A_s(G)) = \mu_G(x)$$

Since the proof will be relevant to a later result, it shall be included (from [3]).

Proof. Let G be a a graph on n vertices, and let G' be the graph obtained by adding loops to each vertex in G. A cycle $v_1, v_2, ..., v_k = v_1$ is simple if $v_1 \neq v_j$ for all $1 \leq i < j \leq k-1$. A disjoint cycle cover of G' is a vertex disjoint collection $\mathcal{C} = \{C_1, ..., C_\ell\}$ of simple cycles (loops and two cycles (single edges) are allowed) in G' such that their union is V. Define the weight of a cycle $C = v_1, ...v_k$ to be

$$w(C) = \begin{cases} x & \text{if } C \text{ is a loop} \\ (-1)^k s(v_1, v_2)s(v_2, v_3)...s(v_{k-1}, v_k) & \text{otherwise} \end{cases}$$

The weight of a disjoint cycle cover $\mathcal{C} = \{C_1, ..., C_\ell\}$ is $w(\mathcal{C}) = w(C_1)...w(C_\ell)$. If we extend s so that $s(i,j) = 0$ whenever $\{i, j\} \notin E$, then

$$\begin{aligned} \det(xI - A_s(G)) &= \sum_{\pi \in \mathcal{S}_n} \text{sign}(\pi) x^{\#\text{ of fixed points}} \prod_{\text{i not fixed}} s(i, \pi(i)) \\ &= \sum_{\mathcal{C}} w(\mathcal{C}) \end{aligned}$$

Note that if C is a cycle of length greater than 2, then $\mathbb{E}w(C) = 0$ because $\{s(e)\}$ are independent and each $s(e)$ has expectation 0. If C is a cycle of length 2, then $w(C) = -1$. Hence

$$\mathbb{E} \sum_{\mathcal{C}} w(\mathcal{C}) = \sum_{\mathcal{C} = \{C_1, ..., C_\ell\}} \mathbb{E}w(C_1)...\mathbb{E}(C_\ell) = \sum_{\mathcal{C}'} w(\mathcal{C}'),$$

where the latter sum is over all disjoint cycle covers into loops and two-cycles. Clearly such cycle covers are in bijection with matchings.

□

This is to say that the expected characteristic polynomial of all signings of a graph is equal to the matching polynomial of the graph. However, though we know that the roots of the expected characteristic polynomial of all signings of the graph, we are not necessarily able to say anything about the roots of the characteristic polynomial of any one signing of the graph. However, Marcus, Spielman, and Srivastava [8] showed that the set of characteristic polynomials of signings of the graph form what they call an **interlacing family** which allowed them to prove the following theorem:

Theorem 4.5. *(Comparison with Expected Polynomial) Suppose* $r_1, ..., r_m \in \mathbb{C}^n$ *are independent random vectors. Then, for every* k*,*

$$\lambda_k(\sum_{i=1}^{m} r_i r_i^*) \leq \lambda_k(\mathbb{E}\chi[\sum_{i=1}^{m} r_i r_i^*](x)),$$

with positive probability, and the same is true with $\geq$ *instead of* $\leq$*.*

We can rephrase signings in terms of independent random vectors, in which case we can apply the theorem and conclude that for a d-regular graph G, there is some signing s such that $\lambda_{max}(A_s) \leq 2\sqrt{d-1}$ and there is some other signing t such that $\lambda_{min}(A_s) \geq -2\sqrt{d-1}$. However, we are not able to say that there exists some signing that meets both of these conditions at the same time.

Nonetheless, we can use Theorem 4.5 in the case where G is a bipartite graph. Bipartite graphs always have symmetrically distributed eigenvalues (that is to say $\lambda \in \mathcal{Spec}(G) \iff -\lambda \in \mathcal{Spec}(G)$). Therefore, in a d-regular bipartite graph, any signing s such that $\lambda_{max}(A_s) \leq 2\sqrt{d-1}$ necessarily also has $\lambda_{min}(A_s) \geq -2\sqrt{d-1}$, so by theorem 4.5, we know that if we start with a bipartite Ramanujan graph (for example, the complete bipartite graph), then at each step there exists a signing s with $\rho(A_s) \leq 2\sqrt{d-1}$ allowing us to create an infinite family of bipartite Ramanujan graphs.

A natural question is to ask how matchings arise in the 2-lift of a graph.

Theorem 4.6. *Let* s *be some signing. Let* $\mathcal{H}^G$ *be the set of all subgraphs* H *of* G *such that every vertex has degree* ≤ 2 *with all even length cycles of* H *containing an even number of edges in* A_s^- *and all odd-length cycles containing an odd number of edges in* A_s^-*. Let* $c(H)$ *be the number of components of* H*,* $\ell(H)$ *be the number of components just consisting of single edges, and* $t(H)$ *be the total number of edges. Let* $\mathcal{H}_k^G = \{H \in \mathcal{H}^G : t(H) \leq k \leq t(H) + \ell(H)\}$*. Then,*

$$m_k(G_s) = \sum_{H \in \mathcal{H}_k^G} 2^{c(H)+t(H)-k} \binom{\ell(H)}{k - t(H)}$$

Proof. Let M be some matching in G_s, and let M_s^{-1} be the edge-induced subgraph of G corresponding to the preimage of M in G. Notice that all vertices in M_s^{-1} must have valency ≤ 2, that is, M_s^{-1} must be composed of paths and cycles. Further, notice that if two edges in M arise from the same edge e in G, then e must be its own component in M_s^{-1}. All other edges map back to just one edge in G.

Let H be a subgraph of G such that the degree of each vertex is ≤ 2. If we have a path $P \in H$ of length k, then the image of P in G_s will be two separate paths of length k. By taking every other edge in each of these paths and one edge from each pair of corresponding edges, we get two different k matchings just from P. If we have a cycle $C \in H$ of length k with an odd number of edges in A_s^-, then C maps to a cycle of length $2k$ in G_s. If k is odd, then by taking every other edge of this cycle (there are two possible ways to do this), we obtain a k matching in G_s. If k is even, then taking every other edge in the image of C will consist of $\frac{k}{2}$ pairs of corresponding edges whose preimage will be every other edge in C; thus in this case, C cannot be in the preimage of any matching.

Now we consider the case where C has an even number of edges in A_s^-. In this case, C maps to two k cycles in G_s. If k is odd, there is no way to choose k edges in the image of C such that one edge from each pair of corresponding edges is chosen, so C cannot be in the preimage of any matching in G_s. On the other hand, if k is even, then there are two ways we can choose one edge from each pair of corresponding edges to get a k matching in G_s. Thus all preimages of matchings in G_s consist of subgraph of G such that each vertex has degree ≤ 2, all odd length cycles have an odd number of edges in A_s^-, and all even length cycles have an even number of edges in A_s^-.

Each component of size k in a subgraph H meeting these conditions produces two different k-matchings in G_s, and when $k = 1$, this component could have been the preimage of two different 1-matchings or one 2-matching. Disconnected components map to disconnected components in lifts. Let $\mathcal{H}^G$, $\mathcal{H}_k^G$, $c(H)$, $\ell(H)$, and $t(H)$ be defined as in the theorem. Then if we have $H \in \mathcal{H}^G$ with at least k edges ($k \geq t(H)$) but with $k \leq t(H) + \ell(H)$ (the maximum number of edges in a matching whose preimage is H is the number of edges in components that aren't single edges plus twice the number of components which are single edges, which is equal to $t(H) + \ell(H)$).

The number of k-matchings whose preimage is H is

$$2^{c(H)-\ell(H)} \times 2^{\ell(H)-k+t(H)} \times \binom{\ell(H)}{k-t(H)} = 2^{c(H)+t(H)-k} \binom{\ell(H)}{k-t(H)}$$

$2^{c(H)-\ell(H)}$ determines edges coming from components which are not single edges. $2^{\ell(H)-k+t(H)}$ comes from single edge components in H which only contribute one edge to a matching whose preimage is H. $\binom{\ell(H)}{k-t(H)}$ stems from the $k - t(H)$ single edge components which contribute two edges to a matching whose preimage is H.

□

5 Relatively Self-Complementary Signings

Definition 5.1. *The* ***relative complement*** *of a graph* G *with respect to a subgraph* H *is* $G \backslash H = \{e \in E(G) : e \notin E(H)\}$. *If* $H \cong G \backslash H$, *we say that* H *is* ***relatively self-complementary*** *with respect to* G.

When we have a graph G_t which comes from the 2-lift of some other graph G via the signing t, we get a natural set of signings S_{RSC} such that for $s \in S_{RSC}$, $A_s^+ \cong A_s^-$, that is say that A_s^+ is relatively self-complementary with respect to G_t. Recall that every edge in G becomes two edges in G_t which we designate as corresponding edges.

Proposition 5.2. *Let* G_t *be the 2-lift of the graph* G *via the signing* t. *Consider signings* s *of* G_t *such that* A_s^+ *contains exactly one edge from each pair of corresponding edges. Then* $A_s^+ \cong A_s^-$, *and the eigenvalues of* A_s *are symmetrically distributed.*

Proof. Recall that G_t contains two copies of the vertex set $V(G)$ which we shall designate as V and V'. Notice that for signings chosen in the way described in the proposition, the permutation $\sigma = (v_1\ v_1')(v_2\ v_2')...(v_n\ v_n')$ is an isomorphism between A_s^+ and A_s^-, so these two subgraphs are isomorphic. Further, notice that σ is order 2, which is to say it is its own inverse. Let P be the permutation matrix associated with σ. We know $P = P^{-1}$, so $P^{-1}A_s^+P = A_s^-$ and $P^{-1}A_s^-P = A_s^+$. Thus

$$P^{-1}A_sP = P^{-1}(A_s^+ - A_s^-)P = P^{-1}A_s^+P - P^{-1}A_s^-P = A_s^- - A_s^+ = -A_s$$

Since conjugation does not change the spectrum of a matrix, the eigenvalues of A_s must be symmetrically distributed. □

As in the case of the proof by Marcus, Spielman, and Srivastava about the existence of Ramanujan 2-lifts for bipartite graphs, looking at the expected characteristic polynomial of a set of signings may tell us about the properties of some single signing in that set. The expected characteristic polynomial of all signings was the matching polynomial, but the proof must be modified to deal with the case of relatively self-complementary signings since the edges are now signed in pairs instead of individually.

Theorem 5.3. *Let* G_t *be the 2-lift of the graph* G *based on the signing* t. *Let* S_{RSC} *be the set of relatively self-complementary signings of* G_t *as described in Proposition 5.2. Let* $\mathcal{C}$ *be as in the proof of Theorem 4.4, and let* $\mathcal{C}'$ *be the set of elements in* $\mathcal{C}$ *composed of loops, single edges (two cycles), and cycles consisting of pairs of corresponding edges (that is, the set of edges* *contained in cycles of length* ≥ 3 *consists entirely of pairs of corresponding edges). Then,*

$$\mathbb{E}_{s \in S_{RSC}} det(xI - A_s(G_t)) = \sum_{\mathcal{C}'} w(\mathcal{C}')$$

Proof. The proof is basically the same as the proof of Theorem 4.4. Each pair of corresponding edges is signed independently from every other pair, so the only non-vanishing terms in the expectation are cycle covers composed of loops and edges (just as in Theorem 4.4) as well as cycles composed of corresponding edges, whose signings are dependent and thus non-vanishing. □

6 Future Work

Interestingly, in all but a few cases tested, the expected characteristic polynomial of relatively self-complementary signings stemming from a Ramanujan 2-lift of a graph was real-rooted. The few cases where this did not hold were when $d = 3$ (and in that case it was uncommon). It is computationally infeasible to calculate expected characteristic polynomials on graphs with more than around 40 edges (number of calculations is $O(2^{\frac{m}{2}})$, so only graphs with $d = 3, 4, 5$ have been tested. Seeing as one hallmark of Ramanujan graph is that they satisfy a reformation of the Riemann hypothesis in terms of the Ihara zeta functions for graphs [7], which is a measure of the distribution of cycles in a graph, we suspect that the Ihara zeta function is intricately linked to the expected characteristic polynomial of relatively self-complementary signings. Future work will involve exploring those connections.

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