Introduction

The space $M_{0,n}$ of configurations of $n$ distinct points on $\mathbb{P}^1$ is a ubiquitous object in algebraic geometry. Hassett produces a family $\overline{M}_{0,\omega}$ of compactifications of $M_{0,n}$ by allowing points to collide in varying degrees, based on their “weight” $\omega$.

A graph associahedron $\mathcal{P}G$ is a convex polytope induced by a finite graph $G$. There is a toric variety $X(\mathcal{P}G)$ associated to $\mathcal{P}G$, and an interesting question with combinatorial and geometric implications is for which $G$, $X(\mathcal{P}G)$ is isomorphic to a Hassett space.

A classic example is the Losev–Manin compactification of $M_{0,n}$, which is given by $\omega = (1,1,\epsilon,\epsilon,...,\epsilon)$ and is isomorphic to $X(\mathcal{P}K_{n-2})$, the toric variety associated to graph associahedron of the complete graph on $n−2$ vertices.

Results

Main Theorem: If $X(\mathcal{P}G) = \overline{M}_{0,\omega}$ for some $\omega$, then $G = \operatorname{Cone}^n\overline{M}_{0,\omega}$ for $\omega = (1,1/2,1/2,\epsilon,\epsilon,...,\epsilon)$.

Graph associahedra

Given a graph $G$ on $n−2$ vertices, fix a bijection between the vertices of $G$ and the facets of the $(n−3)$-simplex $\Delta$.

Definition: A tube $t$ of $G$ is a subset of the vertices of $G$ with connected induced subgraph. $t$ naturally corresponds to the face of $\Delta$ given by the intersection of the facets in bijection to the vertices in $t$.

Construction of $\mathcal{P}G$: Find all tubes of $G$ and truncate the corresponding faces of $\Delta$ in increasing order of dimension.

Hassett spaces

Hassett defines $\overline{M}_{0,\omega}$ by assigning weights to the $\omega$ marked points. Heuristically, “light points” may collide, while “heavy points” may not.

Definition: Given a weight vector $\omega = (\omega_0,\cdots,\omega_n) \in [0,1]^n$, $\overline{M}_{0,\omega}$ is the space of all genus zero rational nodal curves, all of whose components are $\omega$-stable. A component $\Lambda$ is $\omega$-stable if:

1. \# of nodes + $\sum_{i \in \Lambda} c_i > 2$

A Hassett space is a compactification of $M_{0,n}$ of the form $\overline{M}_{0,\omega}$ for some $\omega$.

Examples

If $G$ is a star graph, then $X(\mathcal{P}G) \simeq \overline{M}_{0,\omega}$ for $\omega = (1,1/2,1/2,\epsilon,\epsilon,...,\epsilon)$.

The proof of the main theorem is based on the following observations.

Proposition: If $G$ is a graph, then there exists a weight vector $\omega$ such that $\overline{M}_{0,\omega}$ is isomorphic to $X(\mathcal{P}G)$ iff the following inequalites can be satisfied:

1. for every tube $t$, $|t| > 1, c_{NT} + \sum_{i \in t} c_i > 1$
2. for every non-tube $v$, $c_{NT} + \sum_{i \in v} c_i < 1$
3. $\frac{\sum_{i=1}^{n-2} c_i}{c_i} \leq 1$, $\forall \exists \not\equiv c_m, c_{NT}$

Obstruction 1: If $X(\mathcal{P}G)$ is a Hassett space, no non-tube $v \subset G$ contains a tube $t$ with $|t| > 1$.

Obstruction 2: If $X(\mathcal{P}G)$ is a Hassett space, there is no $S \subset V(G)$ s.t. $S$ can be partitioned in $k$ tubes and $k'$ non-tubes with $k \geq k'$.

Corollary (of obstruction 2): If $X(\mathcal{P}\operatorname{Susp}(G))$ is a Hassett space, $G$ is the complete graph.