Cones of Divisors and Simplicial Complexes

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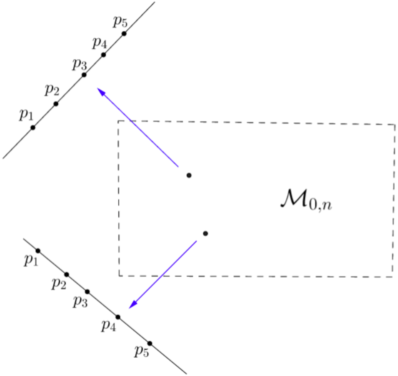
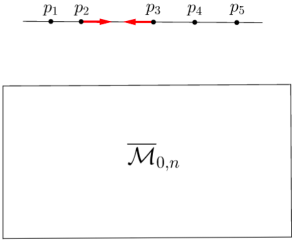
Our Ambient Space: $\overline{\mathcal{M}}_{0,n}$

$\mathcal{M}_{0,n}$ is a space parametrizing all possible configurations of n distinct points on $\mathbb{P}^1$.

$\overline{\mathcal{M}}_{0,n}$ is a space parametrizing all these and all their limits.





Divisors

- Divisors are formal linear combinations of codimension-1 objects $D_1, D_2, \ldots, D_r$ with real coefficients $a_1, a_2, \ldots, a_r$:

$$a_1 D_1 + a_2 D_2 + \cdots + a_r D_r$$

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- After taking the quotient of the space with respect to our equivalence relation, we get a finite-dimensional vector space, N^1, which is isomorphic to $\mathbb{R}^n$.

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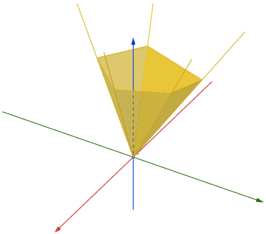
- If $D \in N^1$ is equivalent to an effective divisor, we also say that it is effective.

Cone of Effective Divisors



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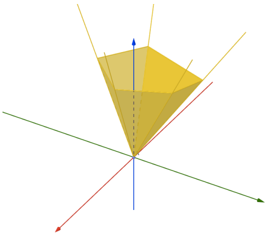

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Doran-Giansiracusa-Jensen recently connected the study of effective divisors on $\overline{\mathcal{M}}_{0,n}$ to the study of certain types of simplicial complexes.

Based on this, we have found new minimal generators of the effective cone of $\overline{\mathcal{M}}_{0,7}$ and in general on $\overline{\mathcal{M}}_{0,n}$.

Simplicial Complexes

- Take $d \in \mathbb{Z}_{\geq 0}$.
- A d-simplex σ is a multiset on $\{1, 2, \ldots, n\}$ such that $|\sigma| = d + 1$. It is nonsingular if there are no repetitions, and is singular otherwise.

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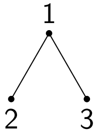
- A d-simplicial complex Δ is a set of d-simplices: $\Delta = \{\sigma_1, \ldots, \sigma_k\}$. It is nonsingular if every simplex it contains is nonsingular, and is singular otherwise.

Visualizing Simplicial Complexes

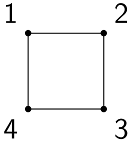
0-complexes are points (or vertices):

$\{\{1\},\{2\}\}$

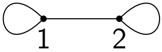
1-complexes can be represented as graphs, with each simplex being an edge:



$\{\{1,2\},\{1,3\}\}$



$\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$

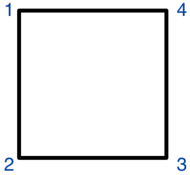


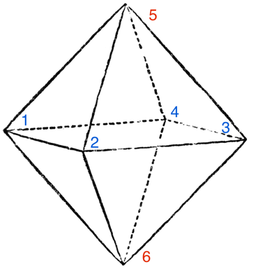
$\{\{1,1\},\{1,2\},\{2,2\}\}$

Product Construction

Given $\Delta_1 = \{\sigma_i\}$ and $\Delta_2 = \{\sigma_j\}$, d_1 and d_2 complex, we define their product as the $(d_1 + d_2 + 1)$-complex

$$\Delta_1 \cdot \Delta_2 = \{\sigma_i \cup \sigma_j : \sigma_i \in \Delta_1, \sigma_j \in \Delta_2\}$$



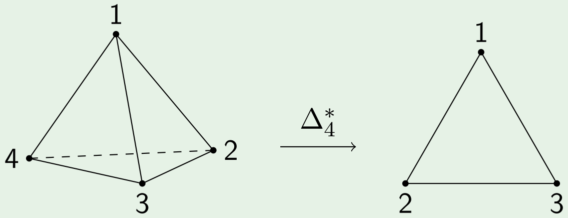


Star Construction at a Vertex

Given Δ with vertex i, we define:

$$\Delta_i^* = \{\sigma \setminus \{i\} : \sigma \in \Delta \text{ s.t. } i \in \sigma\}$$

Example



Weighting

We *weight* a complex over a ring R by assigning each $\sigma_i \in \Delta$ a non-zero value w_i.

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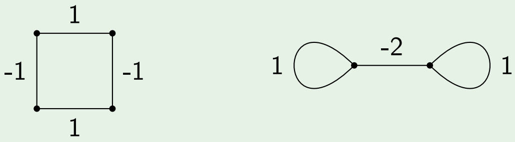


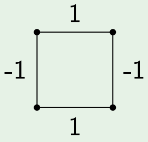
Balancing

A weighted d-complex is *balanced in degree* j if for each multiset S with $|S| = j$ we have:

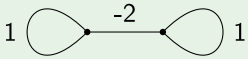
$$\sum_{\sigma_i \supseteq S} w_i \cdot m(S \subseteq \sigma_i) = 0$$

If it is balanced in all $j \in \{0, 1, ..., d\}$, then it is *balanced overall*.

Example



balanced

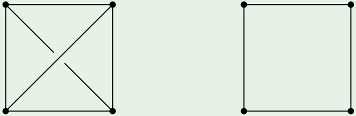


balanced

Minimality

A balanced complex is *minimal* if and only if no proper subcomplex of it is balanceable.

Example



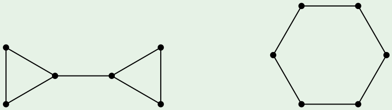
non-minimal minimal

Divisors from Simplicial Complexes

Theorem (DGJ)

If Δ is a balanceable, minimal, nonsingular, nonproduct d-complex on $n \geq d+5$ vertices, it corresponds to a minimal generating divisor D_Δ of $\overline{\mathcal{M}}_{0,n+1}$.

Example



Results

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- Let Δ with a weighting $\{w_i\}$ be a balanced d-complex on n vertices. Then $\forall k \in \{1, ..., n\}$, Δ_k^* is balanceable.
- Let Δ be weighted $\{w_i\}$ on any ring where $(d+1)!$ is not a zero-divisor. If for each $k \in \{1, ..., n\}$, Δ_k^* is balanced with a weighting based on w_i, Δ is balanced.

Computational Results

For indexed multisets $S_i \subset \{1, ..., n\}, i \in \{1, ..., r\}$, and indexed simplices σ_j, we define the $r \times |\Delta|$ multiplicity matrix:

$$M(\Delta)_{ij} = m(S_i \subset \sigma_j).$$

$$M(\Delta) = \begin{array}{c} \\ S_1 \\ \vdots \\ S_r \end{array} \begin{array}{c} \begin{array}{ccc} \sigma_1 & \dots & \sigma_k \end{array} \\ \begin{pmatrix} m(S_1 \subset \sigma_1) & \cdots & m(S_1 \subset \sigma_k) \\ \vdots & \ddots & \vdots \\ m(S_r \subset \sigma_1) & \cdots & m(S_r \subset \sigma_k) \end{pmatrix} \end{array}$$

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Any vector in its kernel corresponds to a balancing weighting of Δ, allowing zeros.

Vector Space of Balancings

The complete singular complex $\Delta_{K,s}$ contains all d-simplicies on n vertices. We have shown:

$$\mathrm{nul}(M(\Delta_{K,s})) = \binom{n+d-1}{d-1}$$

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The complete singular complex $\Delta_{K,s}$ contains all d-simplicies on n vertices. We have shown:

$$\mathrm{nul}(M(\Delta_{K,s})) = \binom{n+d-1}{d-1}$$

The complete nonsingular complex $\Delta_{K,n}$ contains all nonsingular d-simplices on n vertices. We have shown:

$$\mathrm{nul}(M(\Delta_{K,n})) = \binom{n}{d+1} - \binom{n}{d}$$

This implies that there are no nonsingular balanceable d-complexes on $\leq 2d+1$ vertices.

Results

We have used these results, as well as some others, to find classes of complexes that correspond to minimal generating divisors of the effective cone.

- Exhaustively search for them on a given number of vertices
- Triangulating closed manifolds, in particular d-tori.

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- Exhaustively search for them on a given number of vertices
- Triangulating closed manifolds, in particular d-tori.

We have found:

- New minimal generators of the effective cones of $\overline{\mathcal{M}}_{0,7}$.
- Infinitely many new minimal generators of the effective cones of $\overline{\mathcal{M}}_{0,n}$ for large n.

Thank You!

- Jose Gonzalez and Jeremy Usatine
- Nathan Kaplan, Sam Payne, and the Yale Math Department