Combinatorial Brill-Noether theory

Stanislav Atanasov

joint work with Dhruv Ranganathan

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Genus of a graph

Definition

The **genus** of a connected graph $G = (V, E)$, denoted $g(G)$, equals $|E| - |V| + 1$, i.e. the first Betti number of the graph.

Example

The genus of a tree is 0.

Divisors on graphs

A divisor on G is a formal $\mathbb{Z}$-linear combination of its vertices.

Definition

The **degree** of a divisor $D = \sum a_i v_i$ is $\deg(D) = \sum a_i$.

Definition

The divisor $\sum a_i v_i$ is called **effective** if $a_i \geq 0$ for all i.

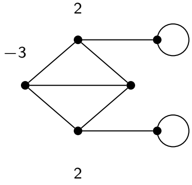

Figure: A non-effective divisor of degree 1.

Equivalent divisors

Vertex **fires** by moving one of its chips to each of its neighbors. This produces a new divisor.

The equivalence relation on divisors generated by chip firing is called **linear equivalence**. The **class** of a divisor D is denoted by $[D]$.

Chip firing: example

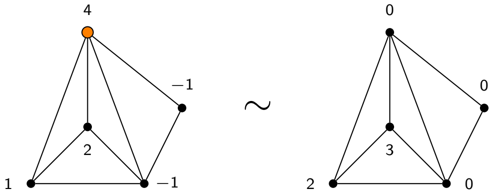

Figure: The orange vertex fires its chips.

Chip firing: example

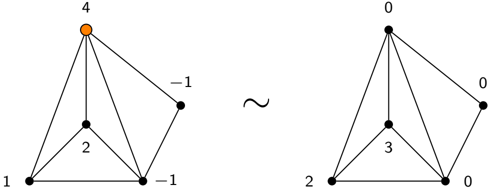

Figure: The orange vertex fires its chips.

Definition

The class $[D]$ is **effective** if it contains an effective divisor.

Ranks of divisors

Definition

The **rank** of D, denoted $\mathrm{rk}(D)$, is the largest integer r such that $[D - E]$ is effective for all effective divisors E of degree r.
If D is not equivalent to effective, then set $\mathrm{rk}(D) = -1$.

Classical Brill-Noether theory

Brill-Noether theory is concerned with the ways in which one can embed an algebraic curve into projective space.

Example

The Brill-Noether existence theorem implies that every algebraic curve of genus g must admit a surjective map f to $\mathbb{P}^1$ of $\deg(f) \leq \lfloor (g+3)/2 \rfloor$.

Combinatorial Brill-Noether theory

Riemann-Roch for graphs (Baker and Norine, 2007)

If D is a divisor on a graph G of genus g, then

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1,$$

where K_G is the **canonical divisor** on G.

Baker's conjectures

Gonality conjecture (Baker, 2008)

Fix $g \geq 0$. Then any graph of genus g has a divisor D with $\deg(D) = \lfloor (g+3)/2 \rfloor$ and $\mathrm{rk}(D) \geq 1$.

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Special case for rank one of the following:

Brill-Noether existence conjecture (Baker, 2008)

Fix $g, r, d \geq 0$, and set $\rho(g, r, d) = g - (r+1)(g - d + r)$. If $\rho(g, r, d) \geq 0$, then for any genus g graph G there exists a divisor D with $\deg(D) = d$ and $\mathrm{rk}(D) \geq r$.

Previous results

Theorem (Baker, 2008)

Brill-Noether existence holds for all graphs of genus at most 3.

Theorem (Cools and Draisma, 2016 (informal statement))

For each homeomorphic class of graphs, there exist edge lengths for which the gonality conjecture holds.

Main result

Theorem (A.,Dhruv R.)

Brill-Noether existence holds for all graphs of genus at most 5.

For $g \leq 5$, the Riemann-Roch theorem and the gonality conjecture imply Brill-Noether existence.

Strategy of proof

- Construct divisors D of $\deg(D) = \lfloor (g+3)/2 \rfloor$ and $\mathrm{rk}(D) \geq 1$ for **topologically trivalent** genus g graphs.

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- Contract edges to obtain constructions for all genus g graphs.

Recall that $\mathrm{rk}(D) \geq 1$ means we can place an *antichip* on any vertex and still chip fire to effective.

Topologically trivalent graphs

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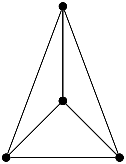

G_2

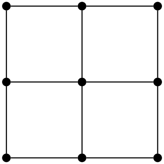

Figure: G_1 is topologically trivalent, while G_2 is not.

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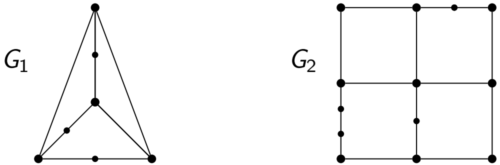

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The gonality conjecture holds for genus g graphs with:

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Figure: Bridges and loops are denoted by dashed lines.

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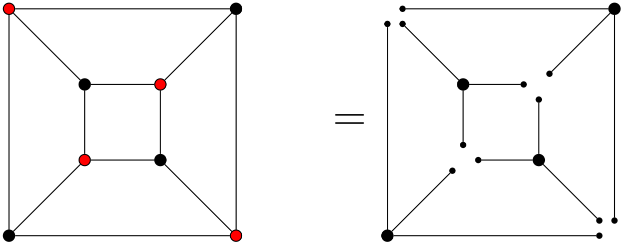
- Treat them on individual basis, but...
- Observe that the support of each divisor of interest divides the graph into a finite collection of connected subgraphs.

Example



We have placed one chip on each red vertex.

From trivalent to general graphs via edge contraction

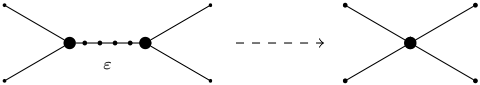


Figure: Edge contraction of the elongated edge ε.

From trivalent to general graphs via edge contraction

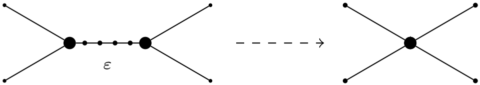
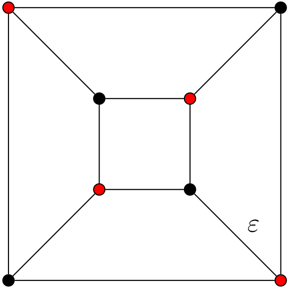


Figure: Edge contraction of the elongated edge ε.

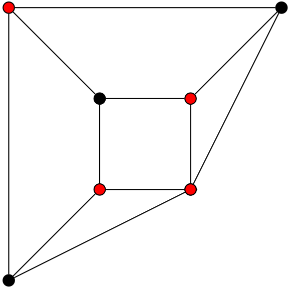
All graphs of genus g are obtained via (repeated) edge contractions of trivalent ones.

From trivalent to general graphs via edge contraction

Note that there is a natural map $\varphi_\varepsilon : Div(G) \to Div(G_\varepsilon)$ for every elongated edge ε of G.



$\xrightarrow{\varphi_\varepsilon}$



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In general NO, but....

In genus 5, the divisor of interest remains of rank at least one under repeated applications of φ_{ε_k} for distinct elongated edges $\varepsilon_1, \cdots, \varepsilon_n$.

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In genus 5, the divisor of interest remains of rank at least one under repeated applications of φ_{ε_k} for distinct elongated edges $\varepsilon_1, \cdots, \varepsilon_n$.

In genus 4, we can only contract one elongated edge ε. The remaining cases are dealt with independently.

Questions?