Non-Left-Orderable Surgeries on Twisted Torus Knots

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Outline

1 Motivation
   - The Boyer–Gordon–Watson L-Space Conjecture
   - Dehn Surgery on Twisted Torus Knots
   - Previous Work

2 Results
   - Non-Left Orderable Surgeries on $T_{p, pk \pm 1}^{\ell, m}$
   - Methodology
   - Outlook
Definition (Left Orderable)

A non-trivial group $G$ is left orderable if it admits a strict total ordering $<$ on its elements that is left invariant, i.e. if $g < h$, then $fg < fh$ for all $f \in G$.

- Ex: $(\mathbb{Z}, +), <$
- Non-Ex: $(\mathbb{Z}_m, +)$
Definitions

Definition (Heegaard Floer Homology)
Heegaard Floer homology is a 3-manifold invariant which associates an $\mathbb{F}_2$-vector space to a closed 3-manifold.

Definition (L-Space)
A closed, connected, orientable 3-manifold is an L-Space if it has the "simplest possible" Heegaard Floer homology.
The Boyer–Gordon–Watson L-Space Conjecture

Conjecture (Boyer–Gordon–Watson)

An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable.

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Twisted Torus Knots
Figure: A right-handed trefoil knot.

Knots (1)
Figure: A figure-eight knot.
Torus Knots (1)

Figure: The (3, 5)-torus knot. (Clay–Watson)
Torus Knots (2)

Figure: The $(3, 5)$-torus knot as a braid.

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Twisted Torus Knots
Twisted Torus Knots

$T_{p,q}^{\ell,m}$ denotes the $(p, q)$-torus knot with $\ell$ strands twisted $m$ full times.

Figure: $T_{3,5}^{2,m}$ (Clay–Watson)
Dehn Surgery

Definition (Dehn Surgery)
Consider a twisted torus knot in $S^3$. Dehn surgery is the process of removing a neighborhood of the knot (a solid torus) from $S^3$ and gluing it back in. This process is specified by a rational number $r$.

Theorem (Vafaee)
Sufficiently large Dehn surgery performed on $T^{\ell,m}_{p,pk \pm 1}$ yields an L-space for either (1) $\ell = p - 1$ or (2) $m = 1$ and $\ell = p - 2$ or (3) $m = 1$ and $\ell = 2$.

Let $G^{\ell,m}_{p,q}(r)$ denote the fundamental group of the manifold that results from $r$-surgery on $T^{\ell,m}_{p,q}$.

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Twisted Torus Knots
Theorem (Clay–Watson)

$G_{3,3k+2}^{2,1}(r)$ and $G_{3,5}^{2,m}(r)$ are not left-orderable for sufficiently large $r$.

- Proof involves case-by-case analysis of generator signs
- Sub-cases of Case 1 ($G_{p,pk\pm 1}^{p-1,m}(r)$) in Vafaee
Results of Clay–Watson

Theorem (Clay–Watson)

$G^{2,1}_{3,3k+2}(r)$ and $G^{2,m}_{3,5}(r)$ are not left-orderable for sufficiently large $r$.

- Proof involves case-by-case analysis of generator signs
- Sub-cases of Case 1 ($G^{p-1,m}_{p,pk\pm 1}(r)$) in Vafaee
Theorem 1 (KC, JG, LH, SV)

$G_{p, pk\pm 1}^{p-1, m}(r)$ is not left orderable for sufficiently large $r$.

- Generalizes work of Clay–Watson
- Lower bound on $r$ is a generalization of Clay–Watson bound
Results (1)

Theorem 1 (KC, JG, LH, SV)

\( G_{p,pk\pm 1}^{p-1,m}(r) \) is not left orderable for sufficiently large \( r \).

- Generalizes work of Clay–Watson
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Twisted Torus Knots
Theorem 2 (KC, JG, LH, SV)

\[ G_{p, pk \pm 1}^{p-2, 1}(r) \text{ is not left orderable for sufficiently large } r. \]

- Results support the L-Space Conjecture.
Characterizing Left Orderability

**Theorem**

A countable group $G$ is left orderable if and only if it is isomorphic to a subgroup of $\text{Homeo}^+(\mathbb{R})$. 

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Global Fixed Points

Definition (Global Fixed Point)
Let $G$ be a group and let $\Phi : G \to \text{Homeo}^+(\mathbb{R})$ be a group homomorphism. $\Phi$ has a global fixed point if there exists a real number $x$ such that $\Phi(g)x = x$ for all $g \in G$.

Proposition (Boyer–Rolfson–Wiest)
If there exists such $\Phi$ with non-trivial image, then there exists another such homomorphism which induces an action on $\mathbb{R}$ with no global fixed points.

- Suffices to show that every homomorphism $\Phi : G_{\ell,m}^{p,pk\pm 1}(r) \to \text{Homeo}^+(\mathbb{R})$ has a global fixed point.
Outlook

- Lower bound on $r$ in our results is larger than the lower bound on surgeries that yield L-spaces.
- The third case of L-spaces described by Vafaee ($m = 1$ and $\ell = 2$) remains unresolved.