# Complex Ramsey Theory

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# Table of Contents

# Arithmetic and Geometric Progressions

**Definition**

A $3$-**term arithmetic progression** is a sequence of natural numbers of the form $(x, x+n, x+2n)$ where $n$ is a positive integer.

**Definitions**

A $3$-**term geometric progression** is a sequence of natural numbers of the form $(x, xr, xr^2)$ where $r > 1$ is an integer. We refer to $r$ as the **common ratio** of the sequence.

# Definitions

## Asymptotic Density

The **density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n}$$

if this limit exists.

## Upper Asymptotic Density

The **upper density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n}.$$

# Table of Contents

# Rankin's Greedy Set

Rankin constructed and characterized a "greedy set" that avoids any $3$-term geometric progressions.

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1 2 3 ~~4~~ 5 6

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1 2 3 ~~4~~ 5 6 7

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1 2 3 ~~4~~ 5 6 7 8

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1 2 3 ~~4~~ 5 6 7 8 ~~9~~

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1 2 3 ~~4~~ 5 6 7 8 ~~9~~ 10

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1 2 3 ~~4~~ 5 6 7 8 ~~9~~ 10 11

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$$\prod_p \frac{p-1}{p} \prod_{i=1}^{\infty} \left(1 + \frac{1}{p^{3^i}}\right) = \frac{1}{\zeta(2)} \prod_{i=1}^{\infty} \frac{\zeta(3^i)}{\zeta(2 \cdot 3^i)} \approx 0.72.$$

# Table of Contents

# Onto the Gaussian Integers

## Definition

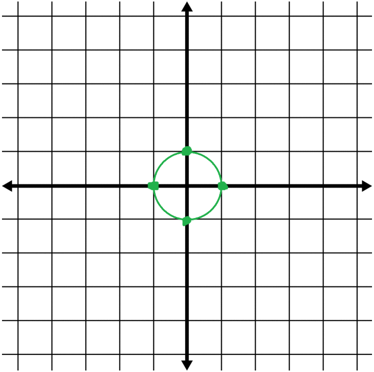
The **Gaussian integers** are defined to be the set of all $a + bi$, where $a$ and $b$ are integers.

## Definition

The **norm** of a Gaussian integer $a + bi$ is defined to be

$$N(a + bi) = a^2 + b^2$$

# Defining the Greedy Set



The greedy set is defined by consideration of “norm circles” whose radii increase.

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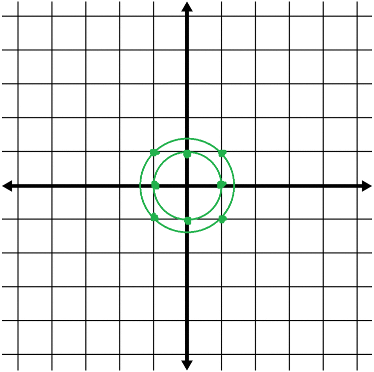

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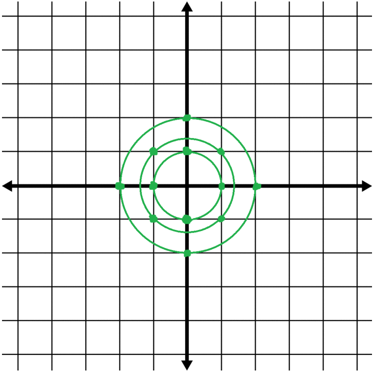

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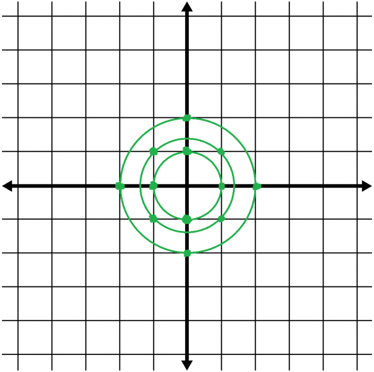
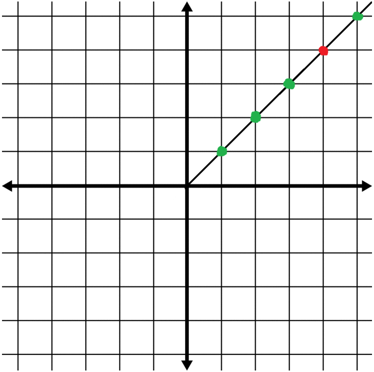


The greedy set is defined by consideration of “norm circles” whose radii increase.

Having defined it, we consider geometric progressions which avoid various kinds of ratios.

# Avoiding Integral Ratios



This case can be thought of as a projection of the integral greedy set onto every line through the origin.

Depicted is the progression $1 + i, 2 + 2i, 4 + 4i$.

# Avoiding Integral Ratios

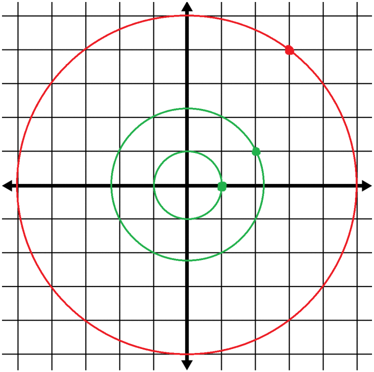
We exclude a Gaussian integer $a + bi$ exactly when it can be written in the form $k(c + di)$, where $k$ is not in Rankin's set and $(c, d) = 1$.

**Theorem 1 [B,H,Mc,Mi,P,T,W '14]**

The density of the greedy set of Gaussian integers that avoids integral ratios is

$$\prod_p \left( \frac{p^2 - 1}{p^2} \prod_{i=0}^{\infty} \left( 1 + \frac{1}{p^{2 \cdot 3^i}} \right) \right) = \frac{1}{\zeta(4)} \prod_{i=1}^{\infty} \frac{\zeta(2 \cdot 3^i)}{\zeta(4 \cdot 3^i)} \approx 0.9397.$$

# Avoiding Gaussian Ratios



We also consider sets that avoid progressions with Gaussian integer ratios.

Depicted is the progression $1, 2+i, 3+4i$.

# Density of the Gaussian Greedy Set

We can determine the likelihood of a Gaussian integer being included by evaluating the primes in its prime factorization and whether each prime is raised to an appropriate power.

**Theorem 2 [B,H,Mc,Mi,P,T,W '14]**

Let $$f(x) = \left(1 - \frac{1}{x}\right) \prod_{i=0}^{\infty} \left(1 + \frac{1}{x^{3^i}}\right).$$

Then the density of the greedy set of Gaussian integers that avoids Gaussian integral ratios is

$$f(2) \left( \prod_{p \equiv 1 \bmod 4} f^2(p) \right) \left( \prod_{q \equiv 3 \bmod 4} f(q^2) \right) \approx 0.771.$$

# Table of Contents

# An Upper Bound for Upper Density

We find an upper bound for the upper density by generalizing an argument by Riddell (1969). Looking at the subset of Gaussian integers with norm $\leq M$, we see

For $b, r \in \mathbb{Z}[i]$ with $N(b) \leq \frac{M}{4}$ and $N(r) = 2$, the terms $b, rb, r^2 b$ have norm $\leq M$ and will always be in geometric progression.

With $N(b)$ odd we know there will be no overlap amongst chosen progressions.

# An Upper Bound for Upper Density

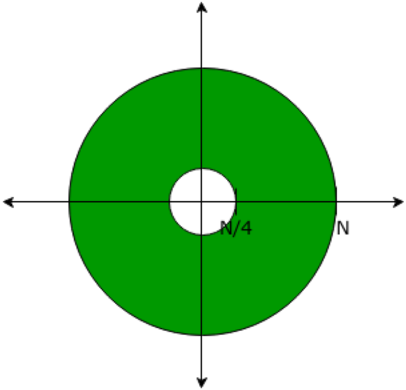
Using Gauss' circle problem, we can exclude about $\frac{1}{2} \cdot \frac{1}{2^2}$ terms. Looking at our next non-overlapping sequence ($r^3b, r^4b, r^5b$ with $N(b) \leq \frac{M}{32}$) and continually repeating this process gives us an upper bound.

**Theorem 3 [B,H,Mc,Mi,P,T,W '14]**

An upper bound for the upper density is given by

$$1 - \frac{1}{2^3} \sum_{n=0}^{\infty} \frac{1}{2^{3n}} = 1 - \frac{\frac{1}{2^3}}{1 - \frac{1}{2^3}} = \frac{6}{7} \approx 0.857$$

# A Lower Bound for Upper Density




Generalizing an argument by McNew, we see that if we take the Gaussian integers with norm between $N/4$ and $N$, no three of these elements will comprise a $3$-term geometric progression.

# A Lower Bound for Upper Density

Similarly, we can include integers with norm between $N/16$ and $N/8$ without introducing a progression, and continue in this fashion.

**Theorem 4 [B,H,Mc,Mi,P,T,W '14]**

A set of acceptable norms is

$$\left(\frac{N}{25}, \frac{N}{20}\right] \cup \left(\frac{N}{16}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

The density of the Gaussian integers that fall inside this set gives us a lower bound of $0.8225$.

# Overview of Bounds

- A lower bound for maximal density of sets of Gaussian integers avoiding integral ratios is $0.9397$.
- A lower bound for maximal density of sets of Gaussian integers avoiding Gaussian ratios is $0.771$.
- Bounds for upper density for sets $S$ of Gaussian integers avoiding Gaussian ratios are $0.8225 < \overline{d}(S) < 0.857$.

# Future Work

- Improve the bounds on upper density for sets $S$ of Gaussian integers avoiding Gaussian ratios.
- Define and analyze the greedy set in other number fields.
- Determine how density of maximal geometric progression-avoiding sets depend on norm and class number of other number fields.

# References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# Thank you!

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