Pseudorandomness of a Markoff Automorphism over $\mathbb{F}_p$

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The Markoff Equation

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Classical Markoff Equation: $x^2 + y^2 + z^2 = 3xyz$

§ 10. -

Disons encore quelques mots sur la résolution en nombres entiers et positifs de l'équation

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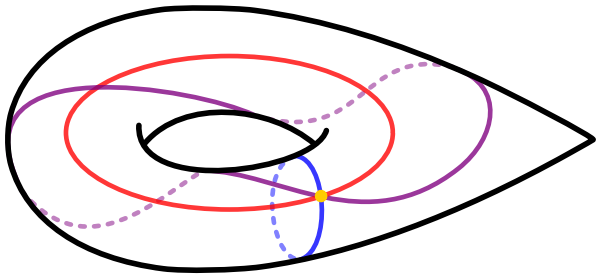
Variant: $x^2 + y^2 + z^2 = xyz$



Solutions over a finite field

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Define the *variety over* $\mathbb{F}_p$

$V(\mathbb{F}_p) = \{(x, y, z) \in (\mathbb{F}_p)^3 \mid x^2 + y^2 + z^2 = xyz\}.$

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We will examine polynomial automorphisms of $V(\mathbb{F}_p)$.

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$x_1 := x,\ x_2 := y,\ x_3 := z;$

$\sigma \in S_3$ acts by $x_i \mapsto x_{\sigma(i)}$.

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Example

$$(1\ 3\ 2) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

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Vieta involutions, even sign changes, and permutations of the coordinates generate the full group Γ of polynomial automorphisms of the variety.

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Strong Approximation: The action of Γ on $V(\mathbb{F}_p) \setminus \{(0,0,0)\}$ is transitive for *all* primes.

Theorem (Bourgain, Gamburd, Sarnak, 2016)

The action of Γ on $V(\mathbb{F}_p) \setminus \{(0,0,0)\}$ is transitive for **almost all** *primes (all but a small and slowly-growing exceptional set).*

Reduced Variety

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Definition

We denote by $H(p)$ the permutation representation of this action.

The remainder of the talk concerns this series $\{H(p)\}$ of finite permutation groups.

Permutation Group, $H(p)$

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Let $|W(\mathbb{F}_p)| = n$.

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$H(p) \leq A_n$ if and only $p \equiv 3 \pmod{16}$.

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Lemma (CGP 2016)

$H(p) \leq A_n$ *if and only* $p \equiv 3 \pmod{16}$.

Conjecture (CGP 2016)

- $H(p) \cong S_n$ for $p \not\equiv 3 \pmod{16}$
- $H(p) \cong A_n$ for $p \equiv 3 \pmod{16}$

We have checked this for primes up to 31.

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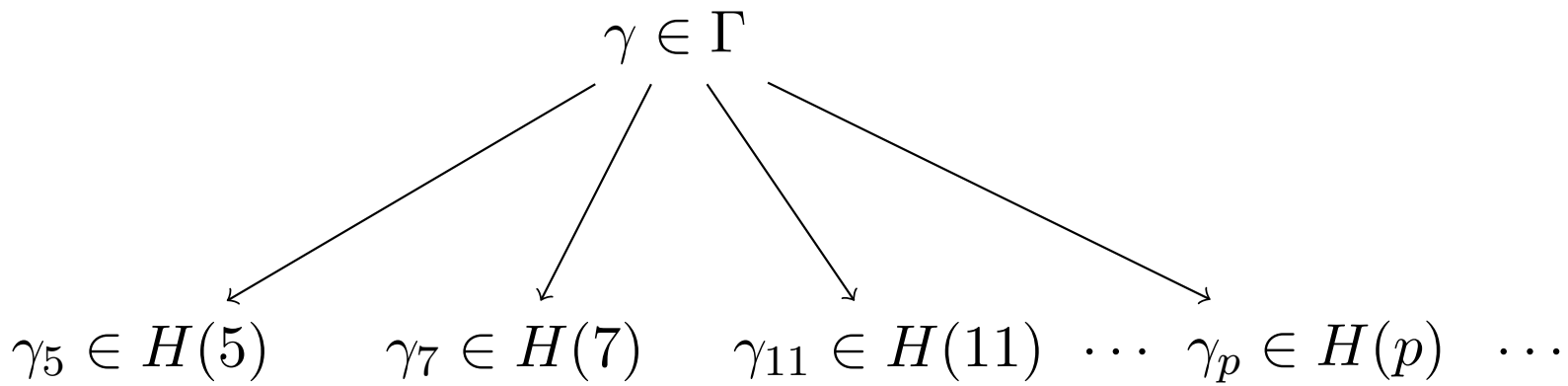

(Recall $H(p)$ is the permutation group generated by the action of Γ on $W(\mathbb{F}_p)$)

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This happens with probability zero.

Fixed Automorphism

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We focus on one element of Γ acting on $W(\mathbb{F}_p)$:

$$\vartheta : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{m_1}{\longmapsto} \begin{pmatrix} yz - x \\ y \\ z \end{pmatrix} \overset{(132)}{\longmapsto} \begin{pmatrix} y \\ z \\ yz - x \end{pmatrix}$$

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This automorphism preserves no obvious structure.

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Fact

For σ chosen uniformly at random from S_n or A_n, and $k < n$, the expected number of k-cycles in the decomposition for σ equals $1/k$.

Goal: Count the number of k-cycles in the decomposition of ϑ_p over all primes.

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For σ chosen uniformly at random from S_n or A_n, and $k < n$, the expected number of k-cycles in the decomposition for σ equals $1/k$.

Goal: Count the number of k-cycles in the decomposition of ϑ_p over all primes. If $\{\vartheta_p\}$ behaves as a family of random permutations, we expect

$$\mathbb{E}_k := \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} \#\{k\text{-cycles in } \vartheta_p\} = \frac{1}{k}$$

where $\pi(x)$ is the prime counting function.

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Theorem (CGP 2016)

Let $\mathbb{E}_k$ be the expected number of k-cycles in the decomposition of $\vartheta_p \in W(\mathbb{F}_p)$. Then

k	1	2	3	4	5
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Proof.

Idea of proofs: For $k \leq 4$, quadratic reciprocity, polynomial manipulations, algebra. For $k = 5$, Galois Theory and Chebotarev Density Theorem. □

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If ϑ_p indeed behaves pseudorandomly, we expect its longest orbits to grow linearly with $|W(\mathbb{F}_p)| \simeq p^2$.

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We prove a lower bound for the size of the largest orbit of the action of ϑ on $W(\mathbb{F}_p)$.

Theorem (CGP 2016)

$\forall$ primes p, $\exists$ an orbit of ϑ on $W(\mathbb{F}_p)$ of length greater than

$$N_0 = \frac{\log(p)}{\log(\varphi)} - \kappa,$$

where κ is a positive constant and $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

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Proof.

Idea: Combinatorial argument, and the fact that a polynomial of degree $n \geq 1$ has no more than n roots.

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Thank you!

Questions