Explicit Bounds for Matrix Pseudospectra

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The Basics

Motivation for Studying Pseudospectra:

Given $A \in \mathbb{C}^{N \times N}$, can ask "Is A singular?"

- The Problem: this is not robust.
- The Solution: Instead ask "Is $\|A^{-1}\|$ large?"

Instead of "Is z an eigenvalue of A?", we ask "Is $\|(z-A)^{-1}\|$ large?"

Definition: Norm of a Matrix

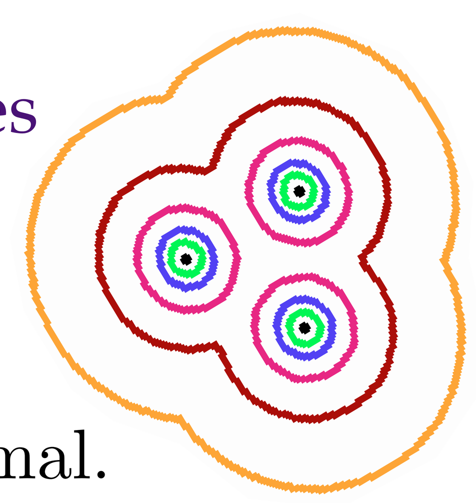
Let A be a matrix, and v a vector. Then, $\|A\| = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|} = \sup_{\|v\|=1} \|Av\|$

Definition: Pseudospectrum of a Matrix

Let $A \in \mathbb{C}^{N \times N}$ and $z \in \mathbb{C}$. We say $z \in \sigma_\epsilon(A)$ if

- $\|(z-A)^{-1}\| > \epsilon^{-1}$
- $z \in \sigma(A+E)$ where E is a matrix such that $\|E\| < \epsilon$
- there exists a vector v, $\|v\| = 1$, such that $\|(z-A)v\| < \epsilon$.

Pseudospectra of Normal Matrices



Let A $\in \mathbb{C}^{N \times N}$. Then,

$$\sigma(A) + \Delta_\epsilon \subseteq \sigma_\epsilon(A) \qquad \forall \epsilon > 0.$$

If $\|\cdot\| = \|\cdot\|_2$, then

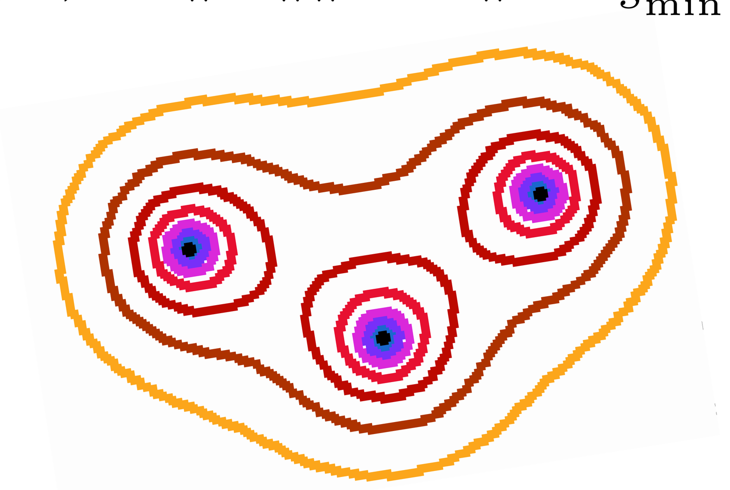
$$\sigma(A) + \Delta_\epsilon = \sigma_\epsilon(A) \Leftrightarrow A \text{ is normal.}$$

Theorem (Bauer-Fike):

Let A $\in \mathbb{C}^{N \times N}$ and let A be diagonalizable, $A = V \Lambda V^{-1}$. Then for each $\epsilon > 0$, with $\|\cdot\| = \|\cdot\|_2$,

$$\sigma(A) + \Delta_\epsilon \subseteq \sigma_\epsilon(A) \subseteq \sigma(A) + \Delta_{\epsilon\kappa(V)}$$

where $\kappa(V) = \|V\|\|V^{-1}\| = \frac{s_{\max}(V)}{s_{\min}(V)}$.



Characterizing Pseudospectra of 2×2 Matrices

Theorem: Non-diagonalizable Matrices

Let A be a non-diagonalizable 2×2 matrix, $A = VJV^{-1}$, where

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, $\sigma_\epsilon(A) = \sigma(A) + \Delta_k$ where $k = \sqrt{\epsilon\alpha + \epsilon^2}$ and $\alpha = \frac{|a|^2+|c|^2}{|ad-bc|}$.

Theorem: Diagonalizable Matrices

Let A be a diagonalizable 2×2 matrix. Then $\sigma_\epsilon(A)$ is the set of points z that satisfy the equation:

$$\left(\epsilon^2 - |z-\lambda_1|^2\right)\left(\epsilon^2 - |z-\lambda_2|^2\right) - \left(\epsilon|\lambda_1 - \lambda_2|\cot\theta\right)^2 < 0,$$

where λ_1, λ_2 are the eigenvalues of A and θ is the angle between the two eigenvectors.

Theorem: Diagonalizable Matrices

Let A be a diagonalizable 2×2 matrix. Then, $\sigma_\epsilon(A)$ asymptotically tends toward a disk. In particular,

$$\frac{r_{\max}}{r_{\min}} = 1 + o(\epsilon),$$

where $r_{\max}$ and $r_{\min}$ are the maximum and minimum distances from an eigenvalue to the boundary of the closer connected component of $\partial\sigma_\epsilon(A)$. Moreover, for A diagonalizable but not normal, $\sigma_\epsilon(A)$ is never a perfect disk.

Pseudospectra of Bidiagonal Matrices

Asymptotic Upper Bound for Resolvent Norm

Let $\lambda_j \in \sigma(A)$ be an eigenvalue of index k_j. Then, asymptotically, $z \in \sigma_\epsilon(A)$ if

$$|z - \lambda_j| \leq (C_j\epsilon)^{\frac{1}{k_j}},$$

where

$$C_j = \|V_j D_j^{k_j-1} U_j^*\| = \|\mathbf{v_{j,1}}\mathbf{u^*_{j,k_j}}\| = \|\mathbf{v_{j,1}}\|\|\mathbf{u_{j,k_j}}\|$$

Case 1

Let A be an $n \times n$ bidiagonal matrix

$$A = \begin{pmatrix} \lambda & b_1 & 0 & 0 & \dots \\ 0 & \lambda & b_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & \lambda & b_{n-1} \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Lower Bound

$$B\left(\lambda, \sqrt[n]{\epsilon(\epsilon+b_1)(\epsilon+b_2)\cdots(\epsilon+b_{n-1})}\right) \subseteq \sigma_\epsilon(A).$$

Asymptotic Upper Bound

$$\sigma_\epsilon(A) \subseteq B\left(\lambda, (|b_1 \cdots b_{n-1}|\epsilon)^{\frac{1}{n}}\right)$$

Case 2

Let A be a $kn \times kn$ matrix with period k, $(a_1, a_2, \dots, a_k)$, on the main diagonal, 1's on the superdiagonal, and 0 elsewhere.

Asymptotic Upper Bound

The asymptotic bound for the resolvent norm of A around each eigenvalue, a_j is given by $\|\mathbf{v_{j,1}}\|\|\mathbf{u_{j,k_j}}\|$, where

$$v_{j,1} = \left(\frac{1}{(a_j-a_{j-1})\cdots(a_j-a_1)}, \frac{1}{(a_j-a_{j-1})\cdots(a_j-a_2)}, \dots, \frac{1}{(a_j-a_{j-1})}, 1, 0, \dots, 0\right)$$

$$u_{j,k_j} = \frac{(a_1-a_j)\cdots(a_{j-1}-a_j)}{[(a_1-a_j)\cdots(a_{j-1}-a_j)(a_{j+1}-a_j)\cdots(a_k-a_j)]^n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (a_{j+1}-a_j)\cdots(a_k-a_j) \\ (a_{j+2}-a_j)\cdots(a_k-a_j) \\ \vdots \\ (a_k-a_j) \\ 1 \end{pmatrix}$$

We can also address the bidiagonal case where the superdiagonal is periodic.

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