Ergodicity of Products in Infinite Measure

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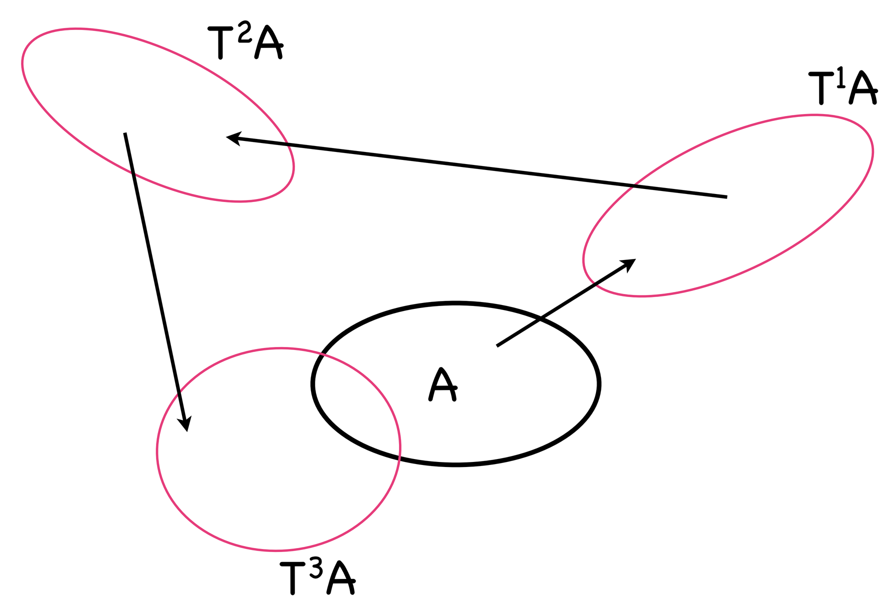
1. Background

- $(X, S(X), \mu)$ a measure space.
- $T : X \to X$ an invertible transformation.
- T is **measure-preserving**, i.e. $\mu(T(A)) = \mu(A)$ for all measurable sets A.

Definition 1.1. *T is* **conservative** *if for any set A of positive measure there exists non-zero integer n such that $\mu(A \cap T^n A) > 0$.*

Figure 1: *Conservativity*



Lemma 1.2. *T is conservative iff for all sets A of positive measure, $\mu\left(A \setminus \bigcup_{n \neq 0} T^n A\right) = 0$.*

Definition 1.3. *T is* **ergodic** *if the only T-invariant sets are X and ϕ, i.e. $TA = A$ implies $A \in \{X, \phi\}$.*

Definition 1.4. *T has* **infinite ergodic index** *if for every $k \in \mathbb{N}$, $\underbrace{T \times \cdots \times T}_{k \text{ times}}$ is ergodic on $\underbrace{X \times \cdots \times X}_{k \text{ times}}$.*

Abstract

We construct a class of rank-one infinite measure-preserving transformations such that for each transformation in the class, all finite cartesian products of the transformation with itself are ergodic but the product of the transformation with it inverse is not ergodic. We also prove that for all rank-one transformations, the product of the transformation with its inverse is conservative.

Significance:

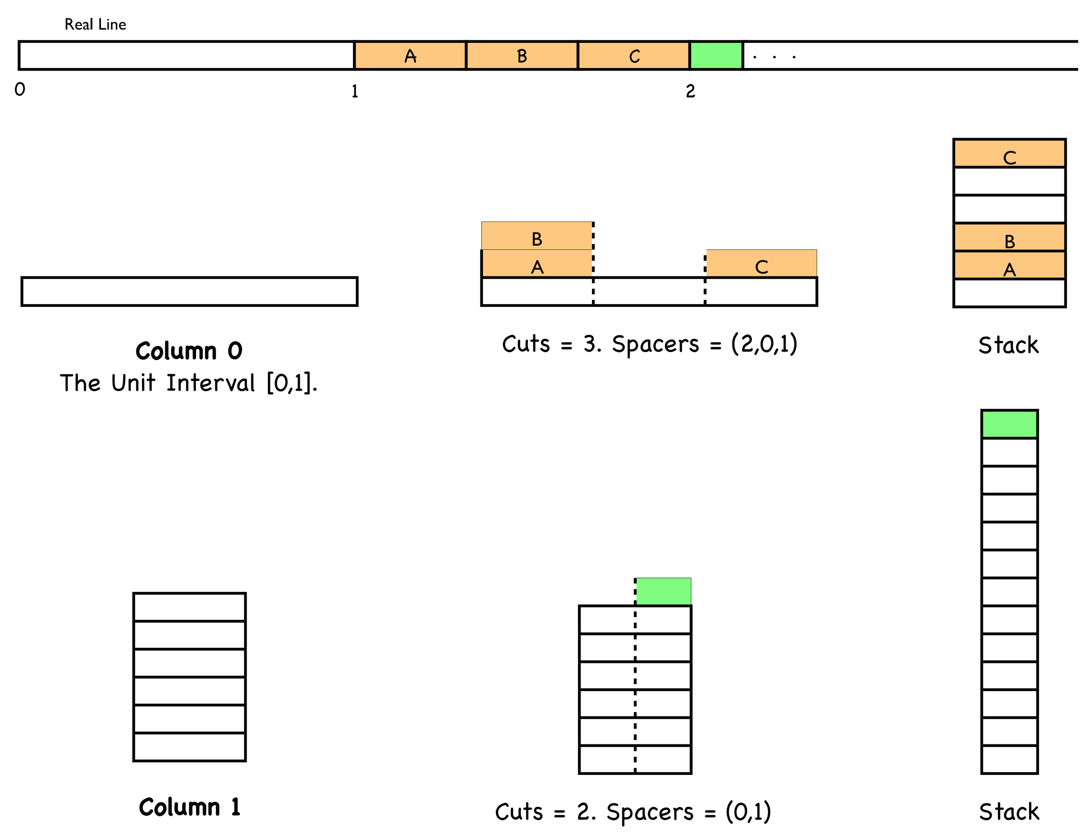
- Brings out the differences in ergodic properties between $T \times T$ and $T \times T^{-1}$ through combinatorial techniques.
- Even the powerful property of infinite ergodic index for T does not force $T \times T^{-1}$ to be ergodic.
- It is well known that $T \times T$ is not always conservative. $T \times T^{-1}$ being conservative for all rank-one T provides interesting contrast.
- Addresses the question of whether T and T^{-1} are isomorphic. Answer is negative because $T \times T$ and $T \times T^{-1}$ do not always agree on conservativity or ergodicity.

2. Rank-One Transformations

2.1 Construction

- Start with a column containing a single level, the unit interval.
- Cut the column into pieces of equal width.
- Add some spacer levels of the correct width from the real line.
- Stack every subcolumn below the one to the right.
- For any point x in a level, let Tx be the point directly above it.
- Repeat the cutting and stacking process indefinitely. Then almost every point has an image under T.

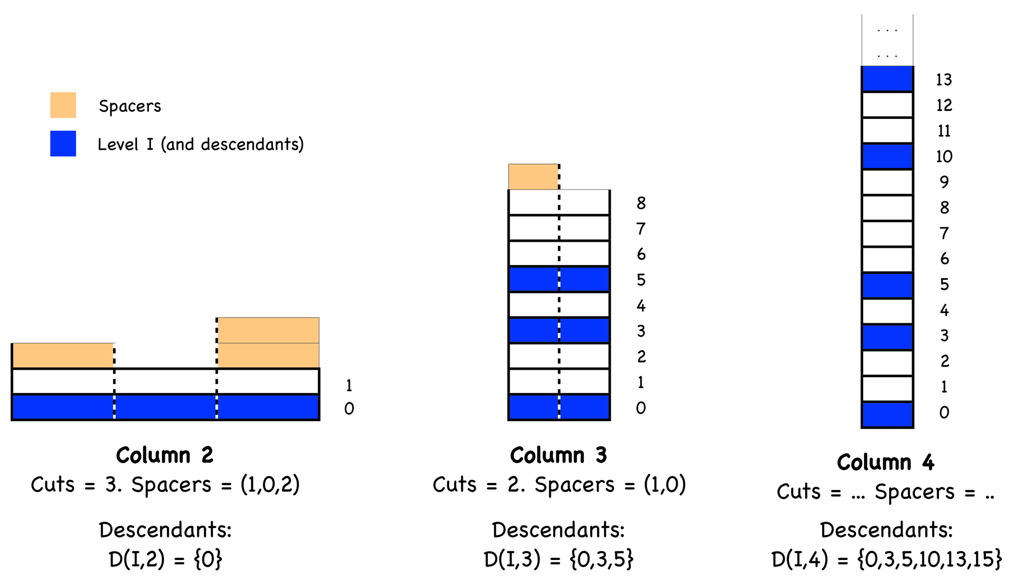
Figure 2: *Construction of Rank-One Transformations*



2.2 Descendants

Definition 2.1. *Let $i \leq j$. Let I be a level in the i-th column. Let J be the base level of the j-th column. Then $D(I, j)$,* **the descendants of I in column** *j, is the set of indices m such that $T^m J \subset I$.*

Figure 3: *Descendants*



Lemma 2.2. *Let I, J be the base levels of the i-th and j-th columns respectively, where $i \leq j$. Then $T^a J \subset T^n I$ iff $a \in n + D(I, j)$.*

This lemma lets us switch from levels to sets of indices, allowing us to invoke combinatorial properties of sets.

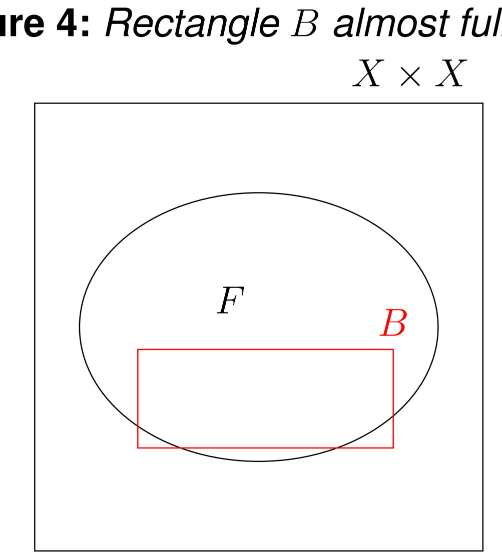
3. Preliminary Work

For the rest of the poster, let $i \geq 0 \in \mathbb{Z}$, let I be the base of column C_i, and let $A = I \times I$.

Strategy:

- We will prove our results on $\mathcal{D}$, the set of all rectangles whose sides are levels in T and T^{-1}.
- We have shown that any property that holds for T on $\mathcal{D}$ must also hold on A
  - Letting $i = 0$, we attain the result for $X \times X$.
- This works because for any set $F \subseteq X$, we can always find some $B \subseteq \mathcal{D}$ such that $\mu(F \cap B) \geq (1 - \epsilon)\mu(B)$, and use our knowledge about B to inform us about F.

Figure 4: *Rectangle B almost full of F*



Lemma: Let T be a rank-one transformation. Then $S := T \times T^{-1}$ is conservative if and only if for every $\epsilon > 0$ there is j such that for at least $(1 - \epsilon)|D(I, j)|^2$ of the pairs $(a_0, a_1) \in D(I, j)^2$, there exists $(d_0, d_1) \in D(I, j)^2$ such that $a_0 + a_1 = d_0 + d_1$, with $a_0 \neq d_0$.

Proof outline: The key to this proof is recognizing that every pair $(a_0, a_1) \in D(I, j)^2$ corresponds exactly to one rectangle, namely $T^{a_0} J \times (T^{-1})^{a_1} J \in \mathcal{D}$.

- There are exactly $|D(I, j)|^2$ such rectangles. Suppose that for some rectangle $T^{a_0} J \times (T^{-1})^{a_1} J$, we have $(d_0, d_1) \in D(I, j)^2$ such that $a_0 + a_1 = d_0 + d_1$, with $a_0 \neq d_0$.
- Then, we may let $n = a_0 - d_0 \neq 0$, so that $S^n(T^{d_0} J \times (T^{-1})^{d_1} J) = T^{a_0} J \times (T^{-1})^{a_1} J$. Now, we have $T^{a_1} J \times (T^{-1})^{a_2} J \subseteq S^n A$.
- If this condition holds for $(1 - \epsilon)|D(I, j)|^2$ of the pairs (a_0, a_1)- that is, for almost all rectangles with sides that are levels in C_j- then we may show that A, up to measure $\epsilon\mu(A)$, is covered by $\bigcup_{n=-m}^{m} S^n A$ (for $n \neq 0$).
- From here, we may conclude that S is conservative.

4. Proof of Main Result

Theorem: For any rank-one transformation T, $T \times T^{-1}$ is conservative.

Proof: By the above lemma, it suffices to show that for every $\epsilon > 0$ there is j such that with probability at least $1 - \epsilon$, a pair $(a_0, a_1) \in D(I, j)$ has a corresponding pair $(d_0, d_1) \in D(I, j)$ such that $a_0 \neq d_0$ and $a_0 + a_1 = d_0 + d_1$.
Suppose that $a_0 \neq a_1$. Let $d_0 = a_1$ and $d_1 = a_0$. Then $d_0 \neq a_0$ and $d_0 + d_1 = a_0 + a_1$, as required. The number of pairs such that $a_0 = a_1$ is $|D(I, j)|$, hence the probability that a pair (a_0, a_1) has a corresponding pair (d_0, d_1) is at least

$$1 - \frac{|D(I, j)|}{|D(I, j)|^2}$$

and this quantity goes to 1 as $j \to \infty$, which concludes the proof.

5. Additional Work

Given a rank-one transformation, we can find the set of descendants, but we can also begin with the set of descendants to define a rank-one transformation. Using this method, we have constructed an example of a rank-one transformation T such that T has infinite ergodic index, but where $T \times T^{-1}$ is not ergodic.

6. Acknowledgements

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