Numerical Semigroups and their Corresponding Core Partitions

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Joint with Hannah Constantin

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Definition
A set $S$ is a **numerical semigroup** if

- $S \subseteq \mathbb{N}$
- $0 \in S$
- $S$ is closed under addition
- $\mathbb{N} \setminus S$ is finite
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$S = \langle 3, 8 \rangle$
Background and Review

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**Example**
$S = \langle 3, 8 \rangle = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, \ldots \}$
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There is an injective map $\varphi$ from numerical semigroups to integer partitions

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$\varphi(S) = (7, 5, 3, 2, 2, 1, 1)$
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Background

**Definition**
A partition $\lambda$ is an $a$–core partition if $a$ does not divide any of the hook lengths of $\lambda$. An $(a,b)$–core partition is both an $a$–core and a $b$–core.

Example
$\lambda = (7, 5, 3, 2, 2, 1, 1)$ is a $(3, 8)$–core partition.
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**Theorem (Anderson)**
For coprime $a$ and $b$, the total number of $(a, b)$–core partitions is

\[
\frac{1}{a+b} \binom{a+b}{a}.
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We are interested in counting the subset of $(a, b)$–cores that come from numerical semigroups via the map $\varphi$. 
Proposition
Suppose $\lambda = \varphi(S)$ for some semigroup $S$. Then $\lambda$ is an $(a, b)$–core if and only if $a, b \in S$. 

Example $\lambda = (7, 5, 3, 2, 2, 1, 1)$ is a $(3, 8)$–core and $\lambda = \varphi(S)$ where $S = \langle 3, 8 \rangle = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, \ldots\}$.
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Definition
Given a numerical semigroup $S$, the set of oversemigroups of $S$ is

$$\{T \supseteq S : T \text{ is a numerical semigroup}\}.$$

The cardinality of this set is denoted $O(S)$.
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The number of $(a, b)$–core partitions from numerical semigroups is exactly $O(\langle a, b \rangle)$.
Definition
If $S$ is a numerical semigroup, then the Apéry tuple of $S$ with respect to some $n \in S$ is the tuple $(k_1, k_2, \ldots, k_{n-1})$ such that $nk_i + i$ is the smallest element of $S$ in its residue class (mod $n$) for each $i$.

This tuple is denoted $\text{Ap}'(S, n)$.
Apéry Tuples

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**Example**
If $S = \langle 3, 8 \rangle = \{0, 3, 6, 8, 9, 11, 12, 14, \ldots \}$, then 16 and 8 are the smallest elements of $S$ in their residue classes mod 3, so $\text{Ap}'(S, 3) = (5, 2)$. 
Apéry Tuples

Suppose $S$ is a numerical semigroup with

$$Ap'(S, n) = (k_1, \ldots, k_{n-1}).$$

A tuple $(\ell_1, \ell_2, \ldots, \ell_{n-1})$ is an Apéry tuple of some numerical semigroup $T \supseteq S$ if and only if the following inequalities are satisfied:

- $\ell_i \geq 0, \forall 1 \leq i \leq n-1$
- $\ell_i + \ell_j \geq \ell_i + j, i + j < n$
- $\ell_i + \ell_j + 1 \geq \ell_n - i - j, i + j > n$
- $\ell_i \leq k_i$ for all $i$

Remark: These inequalities define an $n-1$ dimensional polytope in which the integer lattice points correspond exactly with the oversemigroups of $S$. 
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$S = \langle 3, 8 \rangle$ and $\text{Ap}'(S, 3) = (5, 2)$. The relevant polytope is defined by $x \leq 5$, $y \leq 2$, $2x \geq y$, and $2y + 1 \geq x$: 

\[ \text{There are 10 integer lattice points in this polytope, so } O(\langle 3, 8 \rangle) = 10. \]
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Apéry Tuples and Polytopes

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Theorem (Constantin – H.E.)

If $S = \langle 3, 6 \rangle$ then $O(S) = (3k + \ell)(k + 1)$.

Example

$O(\langle 3, 8 \rangle) = O(\langle 3, 6 \cdot 1 + 2 \rangle) = (3 + 2)(1 + 1) = 10$
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Theorem (Constantin – H.E.)

If $S = \langle 3, 6k + \ell \rangle$ then $O(S) = (3k + \ell)(k + 1)$. 
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**Theorem (Constantin – H.E.)**
If $S = \langle 4, 12k + \ell \rangle$ then $O(S) \sim 24k^3$. 
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If \( S = \langle 4, 12k + \ell \rangle \) then \( O(S) \sim 24k^3 \).

In fact, we can find the explicit formula for each \( \ell \):

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( O(S) )</th>
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<tbody>
<tr>
<td>1</td>
<td>( 24k^3 + 30k^2 + 11k + 1 )</td>
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<tr>
<td>3</td>
<td>( 24k^3 + 42k^2 + 23k + 4 )</td>
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<td>5</td>
<td>( 24k^3 + 54k^2 + 39k + 9 )</td>
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<tr>
<td>7</td>
<td>( 24k^3 + 66k^2 + 59k + 17 )</td>
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<tr>
<td>9</td>
<td>( 24k^3 + 78k^2 + 83k + 29 )</td>
</tr>
<tr>
<td>11</td>
<td>( 24k^3 + 90k^2 + 111k + 45 )</td>
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Asymptotic Behavior

Let \( A(a, b) = \binom{a+b}{a} / (a + b) \), the total number of \((a, b)\)-core partitions by Anderson’s theorem.
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Let $A(a, b) = \binom{a+b}{a}/(a+b)$, the total number of $(a, b)$–core partitions by Anderson’s theorem.

Comparing $O(\langle a, b \rangle)$ with $A(a, b)$ in the limit:

$$
\begin{array}{c|c}
  a & \lim_{b \to \infty} O(\langle a, b \rangle) / A(a, b) \\
\end{array}
$$
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$$\begin{array}{c|c}
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2 & 1 \\
\end{array}$$
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3 & 1/2 \\
\hline
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Future work

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 a & \lim_{b \to \infty} O(\langle a, b \rangle) / A(a, b) \\
 2 & 1 \\
 3 & 1/2 \\
 4 & 1/3 \\
\end{array}
\]
In the future we would like to look at
\[ \lim_{b \to \infty} \frac{O(\langle a, b \rangle)}{A(a, b)} \]
for general values of \( a \).

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In the future we would like to look at

$$\lim_{b \to \infty} \frac{O(\langle a, b \rangle)}{A(a, b)}$$

for general values of $a$.

We suspect that as $a \to \infty$, this fraction will decrease to 0, meaning that almost no $(a, b)$-cores come from semigroups in the limit.
Acknowledgments

We would like to thank...

Nathan Kaplan for guiding our research

Flor Orosz Hunziker and Dan Corey for all their help as mentors

Kyle Luh for helping us understand polytopes

The rest of the SUMRY staff and students for creating such a great program
References

