Splines mod m

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1. Spline Basics

2. Special Properties (mod $m$)

3. Characterizations and the Role of Primes

4. Further Research and New Ideas
Spline Basics
Here is a graph with edges labeled with elements of $\mathbb{Z}/27\mathbb{Z}$.
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Can you label the vertices with ring elements $x_1$ and $x_2$ so that their difference is a multiple of 3?
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Can you label the vertices with ring elements $x_1$ and $x_2$ so that their difference is a multiple of 3?

Of course you can!
Here’s one set of vertex labels you might have found:

\[
\begin{pmatrix} 3 \\ 0 \end{pmatrix}
\]

the set of vertex labels \( \begin{pmatrix} 9 \\ 0 \end{pmatrix} \) is a spline on the graph.
Here are some other splines on the same graph:
Minimal generating sets are very helpful when talking about splines mod $m$:

Here is an edge labeled graph.

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Here is a minimal generating set for all splines on the edge labeled graph.
Definition (Spline mod m)

Let $G$ be an edge labeled graph such that the set of edge labels of $G$ is a subset of $\mathbb{Z}/m\mathbb{Z}$. A **spline mod m** is a set of vertex labels in $\mathbb{Z}/m\mathbb{Z}$ that satisfy the following condition:

- if two vertices labeled $x_1$ and $x_2$ are joined by an edge labeled $\ell_1$ then $|x_1 - x_2| \in \langle \ell_1 \rangle$
- We can look for splines on any type of graph.
- We can find splines on graphs labeled with other rings.
- Let’s look at a few examples of some other cool splines.
more splines

an integer spline on a 3-cycle

a polynomial spline on one edge

a spline on $K_4$ in $\mathbb{Z}/30\mathbb{Z}$
Special Properties (mod $m$)
Special Properties of Splines mod $m$

- Finite sets to label with
- Don’t label with 0 or units
- Variability of the modulus
- Generating set size
\[
\begin{pmatrix}
  x_3 \\
  x_2 \\
  x_1 
\end{pmatrix} : x_j \in \mathbb{Z}
\]
\[(x_3) \in \mathbb{Z}, (x_1) \in \mathbb{Z}, (x_2) \in \mathbb{Z}\]
$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} : x_i \in \mathbb{Z}/6\mathbb{Z}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$
$x_1 \rightarrow 2 \rightarrow x_2 \rightarrow 3 \rightarrow x_3$

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} : x_i \in \mathbb{Z}/6\mathbb{Z}
\]

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}
\]

$3 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$
\[
\begin{pmatrix}
  x_2 \\
  x_1 \\
  x_3 
\end{pmatrix} : x_i \in \mathbb{Z}/6\mathbb{Z}
\]

\[
\left\{ \begin{pmatrix}
  1 \\
  1 \\
  1 
\end{pmatrix}, \begin{pmatrix}
  3 \\
  2 \\
  0 
\end{pmatrix}, \begin{pmatrix}
  3 \\
  0 \\
  0 
\end{pmatrix} \right\}
\]

\[
3 \begin{pmatrix}
  3 \\
  2 \\
  0 
\end{pmatrix} \equiv \begin{pmatrix}
  3 \\
  0 \\
  0 
\end{pmatrix}
\]

\[
\left\{ \begin{pmatrix}
  1 \\
  1 \\
  1 
\end{pmatrix}, \begin{pmatrix}
  3 \\
  2 \\
  0 
\end{pmatrix} \right\}
\]
Our minimal generating sets can be very small.

Theorem (Tymoczko, Hagen)

Let $G$ be an edge labeled graph on $n$ vertices. A minimal generating set for integer splines on $G$ must contain exactly $n$ elements.

Theorem (Tymoczko, Bowden)

Let $G$ be an edge labeled graph on $n$ vertices. A minimal generating set for splines mod $m$ on $G$ can have anywhere between 1 and $n$ elements.*
Generating sets are important and they sometimes behave in surprising ways.

Linear independence can be tricky!

The value of $m$ matters a lot.
Role of Primes
Let's say we want to find a minimal generating set to describe all splines on this graph mod 25...

\[
\begin{pmatrix}
    x_5 \\
    x_4 \\
    x_3 \\
    x_2 \\
    x_1 
\end{pmatrix} : x_i \in \mathbb{Z}/25\mathbb{Z}
\]
Theorem

Let $p$ be a prime number. If $G$ is a graph on $n$ vertices in $\mathbb{Z}/p^2\mathbb{Z}$, then a minimal generating set for all splines on $G$ is:

$$
\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} p \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} p \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \right\}
$$
Bowden

Splines mod m
How about all splines on this graph in \( \mathbb{Z}/32\mathbb{Z} \)?

\[
\begin{pmatrix}
  x_4 \\
  x_3 \\
  x_2 \\
  x_1
\end{pmatrix} : x_i \in \mathbb{Z}/32\mathbb{Z}
\]
$\mathbb{Z}/p^n\mathbb{Z}$ theorem

**Theorem**

Let $p$ be a prime number. If $C_n$ is a cycle on $n$ vertices in $\mathbb{Z}/p^k\mathbb{Z}$, then $\mathbb{B}$ is a minimal generating set for all splines on $G$ (up to rotation).

\[
\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_1 \\ \ell_1 \\ \vdots \\ \ell_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \ell_i \\ \ell_i \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \ell_{n-2} \\ \ell_{n-2} \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_{n-1} \\ \ell_{n-1} \end{pmatrix} \right\}
\]
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Splines mod m
The Importance of Prime Characterizations

- We are working out a structure theorem that uses the prime factorization of $m$ to understand splines mod $m$ in terms of splines mod $p^k$.
- This gives an algorithm to compute minimal generating sets.
- In this way $\mathbb{Z}/p^k\mathbb{Z}$ lets us understand more complex modules of splines.
Future Research
Future Research

- Investigate the relationship between graphs and subgraphs.
- Continue to explore variations in minimal generating set size.
- Continue to investigate other moduli.
- Explore, in greater detail, the relationship between splines mod m and splines over other rings.
- Describe all splines over $\mathbb{Z}/p^k\mathbb{Z}$ for arbitrary $G$.
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