Counting Solutions to the Welch Equation: \( g^{x-1+c} \equiv x \pmod{p^e} \)

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Introduction

The Welch function \( x \to g^{x-1+c} \) is similar to the discrete exponential function \( x \to g^x \), which is used in the ElGamal signature scheme. Counting fixed points of the Welch function could give a benchmark of the difficulty in finding the inverse, which is thought to be computationally intractable in cryptography.

Dissecting the problem

- Study the fixed points of \( g^{x-1+c} \equiv x \pmod{p^e} \) (call it the Welch equation).
- Change the equation into this function: \( f(x, c) \equiv g^{x-1+c} - x \).
- Identify \((x, c)\) pairs in modulo \( p \) where \( f(x, c) \equiv 0 \pmod{p} \).
- Use Hensel’s lemma to count solution pairs in modulo \( p^e \).

Repetition of function \( f(x, c) \)

**Theorem 1.** Consider \( f(x, c) = g^{x-1+c} - x \), where \( c \in \mathbb{Z} \) and \( p \) is an odd prime. Take \( m = \text{ord}_p(g) \). Fix \( x \), then \( f(x, c) \equiv f(x, c + mp^{e-1}) \pmod{p^e} \).

**Theorem 2.** Consider \( f(x, c) = g^{x-1+c} - x \), where \( c \in \mathbb{Z} \) and \( p \) is an odd prime. Take \( m = \text{ord}_p(g) \). Fix \( x \), then \( f(x, c) \equiv f(x + mp, c) \pmod{p^e} \).

Unique \( c \)

**Lemma 3.** Let \( p \) be an odd prime, and \( g \) be a primitive root of \( p \). Then for each \( x \in \{1, 2, \ldots, p - 1\} \setminus \{1, 2p, \ldots, p^e\} \) there exists a unique \( c \in \{1, 2, \ldots, p^e(p - 1)\} \) that is a solution to \( g^{x-1+c} \equiv x \pmod{p^e} \).

Pathway to the main result

- Theorems 1 and 2 tell us that we can restrict the domain of \( c \) when we fix \( x \), and vice versa.
- Theorem 3 tells us that we need to find the values of \( x \) that have solutions, so we can find an associated \( c \).
- When \( g \) is not a primitive root modulo \( p \), it is difficult to predict values of \( x \) which have solutions modulo \( p \) (see example above). The trick is to construct this set: \( \{f(p, c) \pmod{p} : 1 \leq c \leq m\} \). From this we have the Value Set Theorem.

Main Result: \( m^2p^{e-1} \) pairs of solutions

**Theorem 4.** Let \( p \) be an odd prime, and let \( g \in \{1, 2, \ldots, p - 1\} \). Take \( m = \text{ord}_p(g) \). Then for \( x \in \{1, 2, \ldots, mp\} \), and for \( c \in \{1, 2, \ldots, mp^{e-1}\} \), the number of \((x, c)\) pairs of solutions to \( g^{x-1+c} \equiv x \pmod{p^e} \) is \( m^2p^{e-1} \).

Proof of main result.

First we find the number of solutions modulo \( p \). From the Value Set Theorem we see that the number of solutions modulo \( p \) will be \( m \), if \( x \in \{1, 2, \ldots, p\} \). Extend the domain of \( x \) to \( \{1, 2, \ldots, p+1, \ldots, mp\} \), then the number of solutions will increase by a multiple of \( m \). So the number of solutions modulo \( p \) is \( m^2 \).

We expanded the expression for \( f(x, c) \) using a power series and check its partial derivatives. It turns out that at least one of the partial derivatives is non-zero modulo \( p \). Now we use a multivariable Hensel’s Lemma and observe that there are \( p^e - 1 \) possible ways to lift each solution modulo \( p \) to a solution modulo \( p^e \). Thus, there will be \( m^2p^{e-1} \) pairs of solutions.

Value set

**Theorem 5. (Value Set Theorem)** Let \( p \) be an odd prime, fix \( g \) and let \( m = \text{ord}_p(g) \). Consider all \( x \in \{f(p, c) \pmod{p} : 1 \leq c \leq m\} \).

Then a solution \( c \) exists, which solves \( g^{x-1+c} \equiv x \pmod{p} \).

Other patterns

In the process of counting solutions, we were able to predict other results such as:

- If \( g \) is a primitive root, one solution pair is \( (x, c) = (p^e - 1, p^e(p-1)/2) \).
- If \( x, c \) solves \( g^{x-1+c} \equiv x \pmod{p^e} \), then \( g \) is a primitive root.

Conclusion

Theorems 1 and 2 help us understand how to construct the domains of \( x \) and \( c \). The Value Set Theorem helps us identify the solutions modulo \( p \) for the stated domains. However, it is not easy to determine the solution pairs in an arbitrary domain of \( x \) or \( c \). By using these fixed domains, the number of solutions modulo \( p^e \) will always be \( m^2p^{e-1} \).

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